Logarithmic Spirals based on the Golden Ratio, Golden Square Ratio, Silver Ratio, and Silver Square Ratio

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1. Preface

At the end of June, 2010, the writer newly found out that the combination of the golden rhombus and the silver rhombus enables the plane filling. This fact shows that the Golden Ratio and the Silver Square Ratio are closely related to each other. Earlier in 2009, the writer also discovered the mutually complementary relationship between the Golden Ratio and the Silver Ratio in the 3-dimensional space. Considering this point, this new plane filling seems to indicate that the Golden Ratio and the Silver Square Ratio are mutually complementary to each other in the 2-dimensional plane.

As already known, a logarithmic spiral is created from the golden rectangle whose side lengths are in the Golden Ratio. It is also created from the silver rectangle whose side lengths are in the Silver Ratio. Such known facts suggest that there could be other kinds of logarithmic spiral based on the Golden Ratio, Golden Square Ratio, Silver Ratio, and Silver Square Ratio. This paper describes the outcome of study on these possibilities.

2. Logarithmic Spiral based on the Golden Ratio, Golden Square Ratio, Silver Ratio, and Silver Square Ratio

The regular pentagon includes two kinds of isosceles triangles. One is the obtuse triangle whose oblique side length and base length are in the ratio of $1:\phi$ and the other is the acute triangle whose oblique side length and base length are in the ratio of $\phi:1$. Here, ϕ represents the Golden Ratio. Therefore, if the isosceles triangle whose side lengths are in the Golden Ratio is defined to be the golden triangle, the following proposition holds water.

<u>Proposition</u> : The regular pentagon is divided by two diagonals into two obtuse isosceles triangles and one acute isosceles triangle.

Sir Roger Penrose, a famous mathematical physicist of Oxford, discovered two kinds of non-periodical plane filling patterns in 1974. One is composed of a sagittal shape and a trapezoidal shape and the other consists of two kinds of rhombus. The isosceles triangle equal to one-half of such sagittal shape and trapezoidal shape respectively corresponds to the aforementioned two kinds of golden triangle.

These two triangles have a distinguishing characteristic. Let's say that the obtuse isosceles triangle and the acute isosceles triangle are temporarily named A and B, respectively. If the largest possible B is removed from A, the rest becomes A. If the largest possible B is further removed from it, the reminder again becomes A. Conversely, if the largest possible A is removed from B, the rest becomes B. If the largest possible A is further removed from it, the reminder again becomes B. This is very similar to a characteristic feature of the golden rectangle; i.e. if a square section is removed, the remainder becomes another golden rectangle; that is, with the same aspect ratio as the first, and such square removal can be repeated infinitely.

Now, let's think of a "silver triangle" which parallels these golden triangles on the basis of the above-mentioned definition. Then, it is immediately found that the isosceles right triangle equal to one-half of the square corresponds to the golden triangle A or B. Accordingly, If the isosceles right triangle whose side lengths are in the ratio of 1 : √2 is defined to be the silver triangle, the following proposition holds water.

Proposition : The square is divided by one diagonal into two congruent silver triangles.

This triangle has a salient feature very comparable to the one of the silver rectangle. Namely, if the triangle is folded in half, it becomes a similar figure. If the half-sized triangle is further folded in half, it again becomes a similar figure, and this process can be repeated infinitely.

As mentioned earlier, the golden rectangle includes the infinite number of squares. If quadrants are continually drawn in such squares with their radius being equal to the square's side length, a curve extremely close to a logarithmic spiral appears as shown in Fig. 1. This is the approximation method described in the writings of Johannes Kepler.

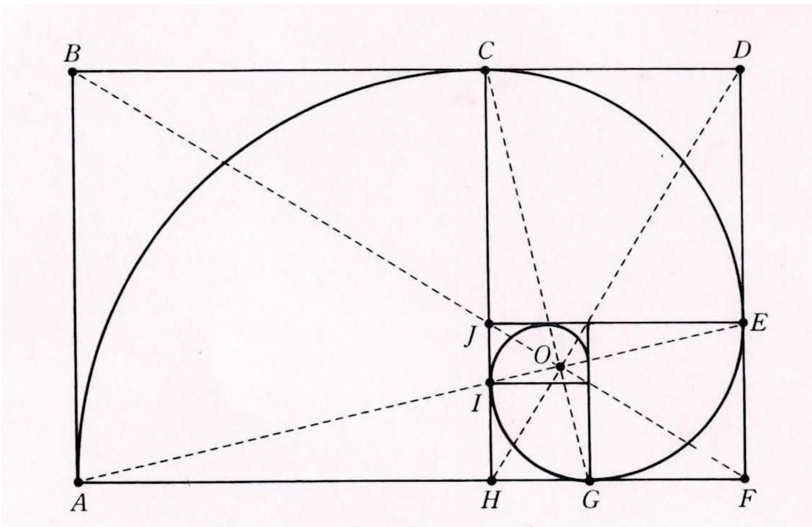


Fig. 1: Approximate construction of logarithmic spiral based on the golden rectangle

In general, a logarithmic spiral is expressed in polar coordinates like $r = B^{\theta}$. The value of B varies depending on the kinds of spiral. Logarithmic spirals are also created from the aforementioned two kinds of golden triangle, golden rhombus, silver rhombus, silver rectangle and silver triangle. These spirals are all expressed by polar equation.

A rectangular parallelepiped has three different faces. In case of the one called "Silver Rectangular Parallelepiped", two of these faces are equal to the silver rectangle and the third face is equal to the silver square rectangle. The latter's two different sides are in the Silver Square Ratio. It is composed of one golden rectangle and one golden square rectangle. Two different sides of the latter rectangle are in the Golden Square Ratio. However, this rectangle consists of one silver square rectangle and one golden rectangle. Such relationships are shown in the following two figures.

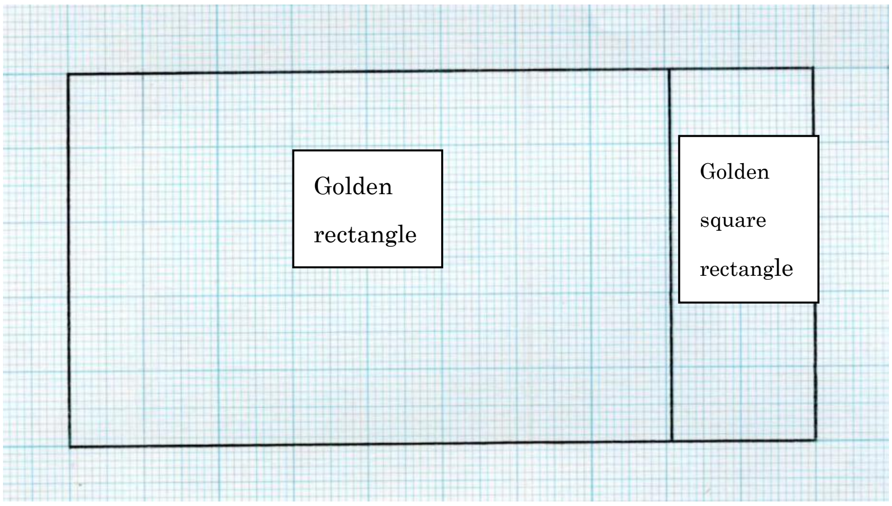


Figure 2: Silver square rectangle composed of one golden rectangle and one golden square rectangle

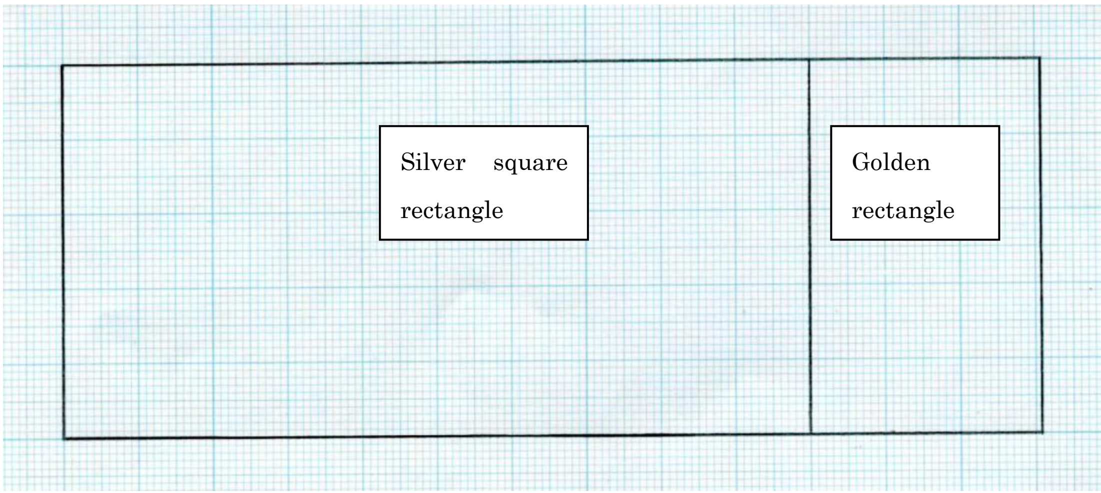


Figure 3: Golden square rectangle composed of one silver square rectangle and one golden rectangle

This means that the silver square rectangle contains the infinite number of golden rectangles. This fact also seems to indicate that the Golden Ratio and the Silver Square Ratio are mutually complementary to each other in the 2-dimensional plane. As shown in Figure 4, golden rectangles line up in sequence alongside of two different flows, filling in the silver square rectangle. It also represents that the golden square rectangle contains the infinite number of golden rectangles.

Two new logarithmic spirals are derived from the silver square rectangle and expressed in the same equation as the aforementioned logarithmic spirals. (Refer to Figure 4.) These spirals are identical and their centers are also the same.

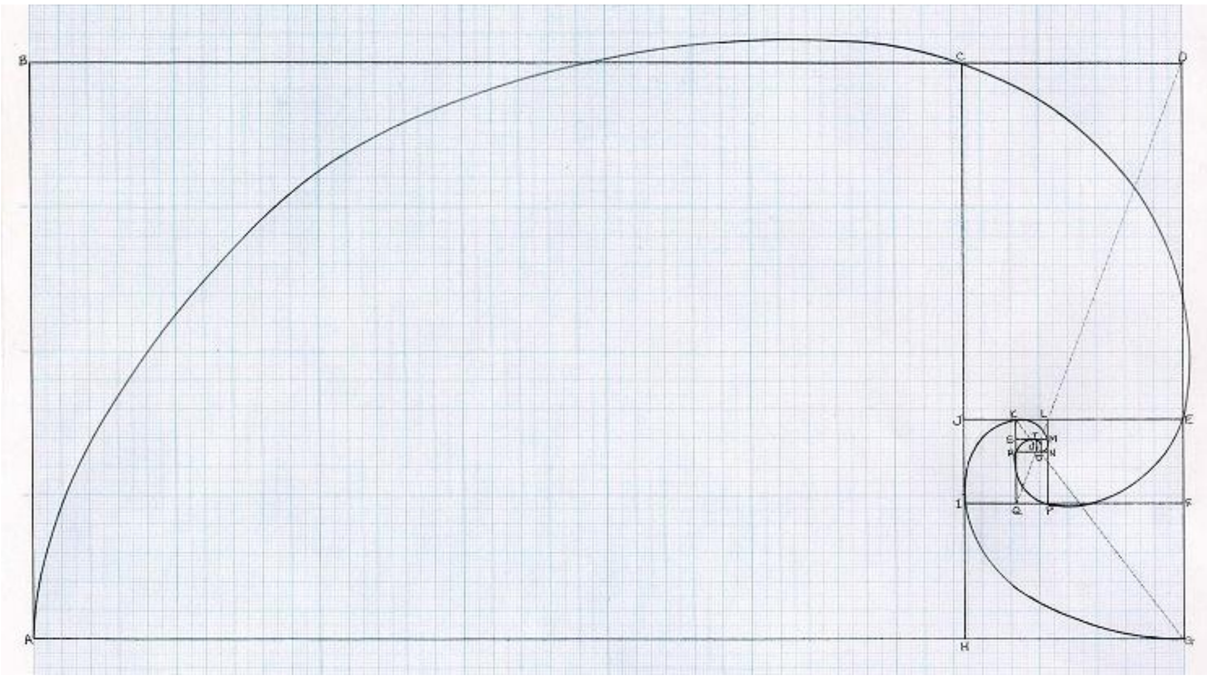


Figure 4: Two new logarithmic spirals based on the silver square rectangle
(The silver square rectangle contains the infinite number of golden rectangles.)

3. Two Different Polar Equations expressing Logarithmic Spirals

Logarithmic spirals are expressed in two different polar equations. One is: $r = B^{\theta}$ as mentioned above and the other is:

$$r = ae^{\theta \cot b}$$

In this equation, a represents the length r when θ equals zero. If ψ is defined to be equal to $\pi - b$, ψ represents the angle between a straight line drawn from the coordinate center and a straight line tangential to a logarithmic spiral. These two equations are equal. Therefore, they make it possible to calculate the value of b and ψ. Mr. Ikuro Sato of the Research Institute, Miyagi Cancer Center, Japan kindly informed the writer of a method for calculating them. A result of such calculation was: b= arctan (1/log B).

Table 1 below shows: (1) the value of B, (2) the reason for it, (3) the value of b, and (4) the value of ψ, for the aforementioned eight kinds of logarithmic spiral.

As shown in this table, the logarithmic spiral derived from the golden rhombus is equal to the one derived from the golden rectangle. This is because the value of B and b are the same in both cases. For the same reason, the logarithmic spiral derived from the silver rhombus is equal to the one derived from the silver rectangle. Accordingly, it can be said that the logarithmic spirals based on the Golden Ratio, Golden Square Ratio, Silver Ratio, and Silver Square Ratio are of six kinds in total.

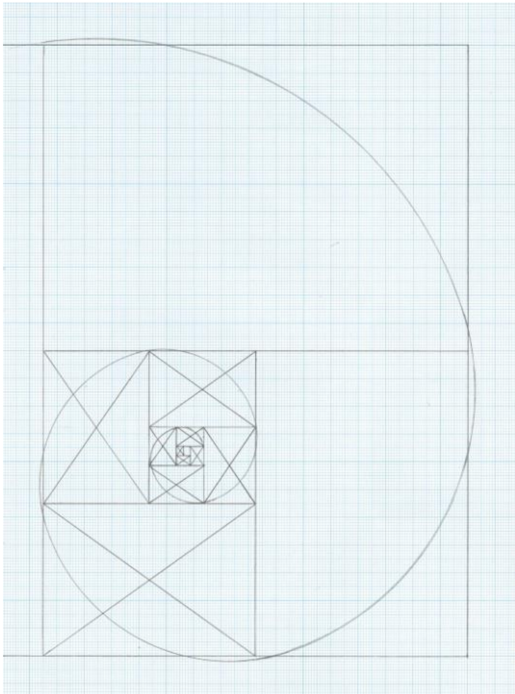
	$r = B^{\theta}$		$r = ae^{\theta \cot b}$	
	B	Reason	b	$\psi = \pi - b$
Spiral derived from golden rectangle	$B = \phi^{\frac{2}{\pi}}$	Value of r increases by a factor of ϕ (golden ratio) every time θ gains by $\pi/2$ (=90°)	0.405376π	0.594624π
Spiral derived from silver rectangle	$B = \sqrt{2}^{\frac{2}{\pi}}$	Value of r increases by a factor of $\sqrt{2}$ (silver ratio) every time θ gains by $\pi/2$ (=90°)	0.430877π	0.569123π
Spiral based on acute golden triangle	$B = \phi^{\frac{5}{3\pi}}$	Value of r increases by a factor of ϕ every time θ gains by $3\pi/5$ (108°)	0.420438π	0.579562π
Spiral derived from obtuse golden triangle	$B = \phi^{2\frac{5}{2\pi}}$	Value of r increases by a factor of squared ϕ every time θ gains by $2\pi/5$ (=72°)	0.2919π	0.7081π
Spiral derived from silver triangle	$B = 2^{\frac{2}{\pi}}$	Value of r increases by a factor of 2 (silver square ratio) every time θ gains by $\pi/2$ (=90°)	0.36772π	0.63228π
Spiral derived from golden rhombus	$B = \phi^{2\frac{1}{\pi}}$	Value of r increases by a factor of squared ϕ every time θ gains by π (=180°)	0.405376π	0.594624π
Spiral derived from silver rhombus	$B = 2^{\frac{1}{\pi}}$	Value of r increases by a factor of 2 (silver square ratio) every time θ gains by π (=180°)	0.430877π	0.569123π
Spiral derived from silver square rectangle & golden square rectangle	$B = \phi^{2\frac{2}{\pi}}$	Value of r increases by a factor of squared ϕ every time θ gains by π (=90°)	0.325π	0.675π

Table 1 The value of B, the reason for it, the value of b, and the value of ψ, for the aforementioned eight kinds of logarithmic spiral.

4. How to Make Approximate Construction of Logarithmic Spiral

Concerning the spirals derived from the golden rectangle and the acute golden triangle, the way to make approximate construction has already been released. It does not appear, however, that such information on the other spirals has been published.

The writer made a lot of study on it and, as a result, such construction method was newly found concerning the other spirals with the exception of the one derived from silver square rectangle & golden square rectangle and the one derived from golden rhombus. It is shown below just for reference. Of such construction, the one of the spiral derived from the acute golden triangle (shown in Figure 10) is not what the writer newly found. It has already been open to the public.



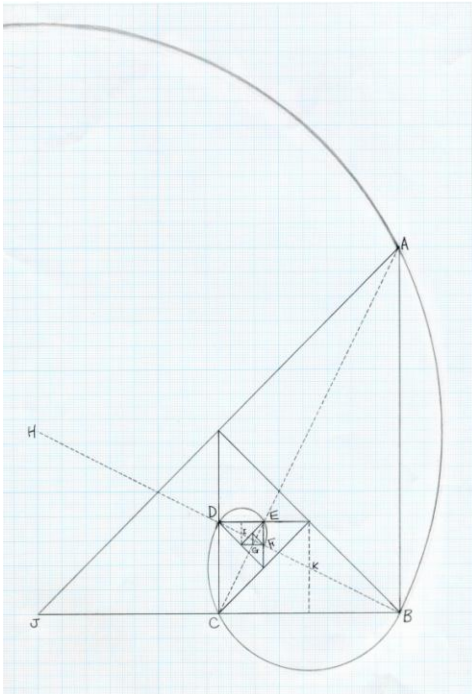
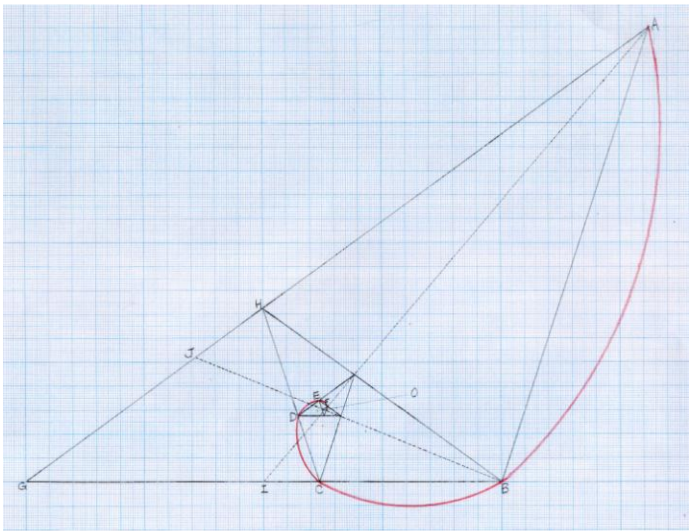


Fig. 5: Approximate construction of logarithmic spiral derived from the silver rectangle

Draw a quarter circle with an intersection of diagonals of a sequence of silver rectangles as a center

Fig. 6: Approximate construction of logarithmic spiral derived from the silver triangle

Draw a quarter circle with a midpoint of hypotenuse of a sequence of silver triangles as a center



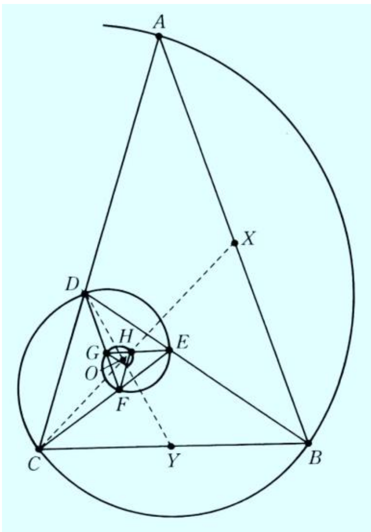


Fig. 7: Approximate construction of logarithmic spiral derived from the obtuse golden triangle

Draw a regular triangle with AB, one of two equal sides of the obtuse golden triangle, as its base. Then, draw a sixth circle with a vertex of the regular triangle as a center. Then, continue this process, making each sixth circle connected.

Fig. 8: Approximate construction of logarithmic spiral derived from the acute golden triangle

Draw an arc of 108° with D, a vertex of *the obtuse golden triangle ADB, as a center. Then, draw such arc with E as a center. Then, continue this process, making each arc connected.*

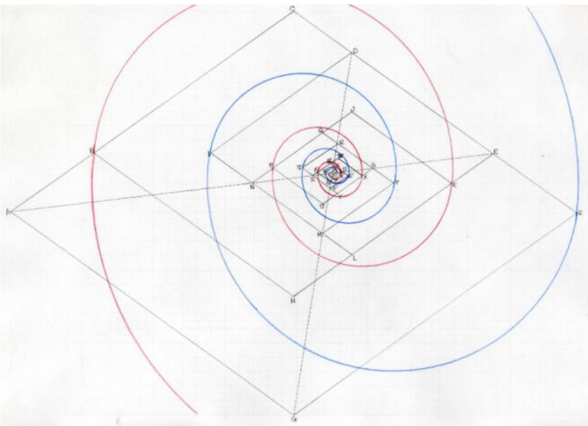


Fig. 9: Approximate construction of logarithmic spiral based on acute angle vertices of the silver rhombus

Draw a semicircle with acute angle vertices of a sequence of the silver rhombus as a canter

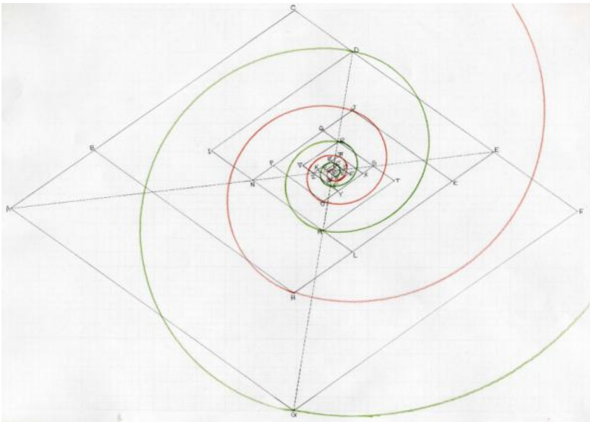


Fig. 10: Approximate construction of logarithmic spiral based on obtuse angle vertices of the silver rhombus

Draw a semicircle with obtuse angle vertices of a sequence of the silver rhombus as a canter

5. Conclusion

- The regular pentagon is divided by two diagonals into two obtuse isosceles triangles and one acute isosceles triangle.
- The square is divided by one diagonal into two congruent silver triangles.
- The silver square rectangle is composed of one golden rectangle and one golden square rectangle. Furthermore, the golden square rectangle consists of one silver square rectangle and one golden rectangle. This means that the silver square rectangle contains the infinite number of golden rectangles and that the golden square rectangle also contains the infinite number of golden rectangles. This fact also seems to indicate that the Golden Ratio and the Silver Square Ratio are mutually complementary to each other in the 2-dimensional plane.
- The golden rectangle, silver rectangle, acute golden triangle, obtuse golden triangle, silver triangle, golden rhombus, silver rhombus, silver square rectangle, golden square rectangle are based on either the Golden Ratio, Silver Ratio, Golden Square Ratio, or Silver Square Ratio. The logarithmic spirals derived from these polygons are of six kinds in total.
- In consideration of the foregoing, it is strongly suggested that the Silver Ratio is, in various aspects, of the same significance as the Golden Ratio.

<u>References</u>

[1] A. Beutelspacher and B. Petri, Golden Section: Nature, Mathematical Principle, and Art, Kyoritsu Shuppan, 2005

[2] K. Miyazaki, “Panorama of Shapes”, Maruzen, 2003