Review

Four New Counter-examples to Kelvin's Conjecture on Minimal Surfaces

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Four new counter-examples to Kelvin's conjecture on minimal surfaces have been found. In 1994 Weaire and Phelan discovered a counter-example to Kelvin's such conjecture whose periodic unit includes two cells of different kind, i.e. the pentagonal dodecahedron and the Goldberg tetrakaidecahedron (hereinafter called WP). It was verified to be more superior to Kelvin's tetrakaidecahedron in terms of isoperimetric quotient. WP already contained minimum curved surfaces when it was discovered with the help of the Surface Evolver. The four counter-examples presented here didn't. Therefore, they were initially compared with WP with its faces having been flattened out for planar approximation. Such comparison revealed that the four counter-examples represent partitions with slightly less surface area than the flattened WP structure. In order to enable equivalent comparison, they were then transformed to have minimum curved surfaces included.

Keywords: Square root of 2 ratio, minimal surface area partition of space, minimum curved surfaces, diagonal, dihedral, Plateau's rules

1. Introduction

Kelvin's conjecture is the 3D version of the classical honeycomb conjecture which was formally proven by Hales in 1999 [1]. It is naturally considered that the honeycomb represents an efficient partition of space with minimal surface area. It is based on the rhombic dodecahedron consisting of 12 rhombuses. This rhombus is very special in that the length ratio of its shorter diagonal to longer diagonal is $1{:}\sqrt{2}$. A regular hexagon composed of three rhombuses shows up when this polyhedron is projected to an X-Y plane with the Z-axis serving as 3-fold rotational symmetry axis. This polyhedron becomes a honeycomb if it is extended in the direction of Z-axis like hexagonal cylinder with its front side opened.

In 1887 the Philosophical Magazine published Kelvin's classic analysis of this problem [2]. This paper is grounded on Plateau's theoretics on soap bubbles [3]. It states as follows:

- Surfaces of soap bubbles always meet in threes along an edge called a Plateau border and they do so at an angle of $\cos^{-1}(-1/2)$, which is equal to $120°$.
- These Plateau borders meet in fours at a vertex and they do so at an angle of $\cos^{-1}(-1/3)$, which is equal to approx. $109.471°$, the tetrahedral angle.

He proposed that the body-centered cubic structure as a probable candidate for the optimal arrangement. What is now well-known as the Voronoi cell of the body-centered cubic structure is the truncated octahedron with 6 squares and 8 regular hexagons. This polyhedron has 2 kinds of dihedral, i.e. $\cos^{-1}(-1/3)$ and $\cos^{-1}(-1/\sqrt{3})$. The former is equal to approx. $109.471°$ and the latter is equal to approx. $125.264°$. While these dihedrals do not follow Plateau's rules, Kelvin successfully satisfied them by marginally deforming the hexagonal faces. This structure is depicted in Fig.1 below.

As a general rule, if A is the average interfacial area per cell and V is the volume of each cell in the periodic partition, a dimensionless, scale-invariant quantity c can be defined as:

$$c = A/V^{\frac{2}{3}} \quad (1.1)$$

This quantity, known as the isoperimetric quotient [5], is used throughout this work for comparison purposes between different structures. Whereas c of the rhombic dodecahedron is equal to approx. 5.345, the one of the truncated octahedron is equal to approx. 5.315. The latter is superior to the former by approx. 0.57%. On the other hand, the slight distortion of the hexagonal faces of the truncated octahedron decreases the total surface area of the cell by approx. 0.17% [4] and, as a result, its isoperimetric quotient was enhanced to approx. 5.306.

This solution K presented by Kelvin was believed to be most superior till 1994, when Denis Weaire and Robert Phelan indicated the existence of a new partition WP with less surface area than K [6]. They utilized Ken Brakke's computer program, Surface Evolver [7], which simulates the least surface area of cells. This partition known as the Weaire-Phelan structure is predicated on the combination of two different kinds of cell, i.e. the pentagonal dodecahedron and the Goldberg tetrakaidecahedron. The pentagonal dodecahedron consists of 12 pentagons and the tetrakaidecahedron is composed of 2 hexagons, 4 pentagons which are the same as those of the pentagonal dodecahedron, and 8 smaller pentagons. None of them are regular polygons. All pentagons, larger and smaller, have slight curvature. The isoperimetric quotient c of WP stands at approx. 5.288 [6]. It is superior to K in terms of the isoperimetric quotient by approx. 0.34% and, therefore, has been regarded as being the optimum partition of space since then.

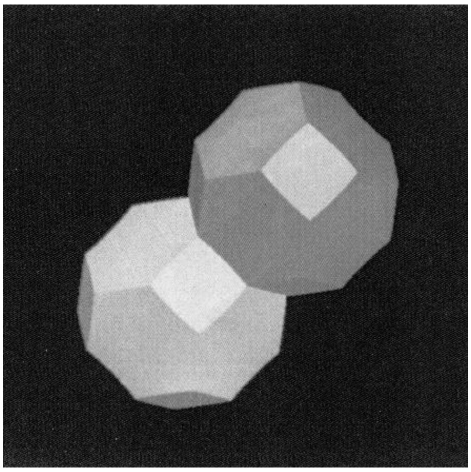

Fig.1 Kelvin's tetrakaidecahedron

2. Six Salient Features of Partition based on the Combination of Polyhedra composed only of Pentagons and Hexagons

Soon after the discovery of WP, John Sullivan indicated that there exist infinite kinds of partition predicated on the combination of polyhedra consisting only of pentagons and hexagons, which are known as tetrahedrally close-packed structures [9]. These include combinations of dodecahedra that are composed only of one kind of pentagon and

Goldberg tetrakaidecahedra that are made up of 2 kinds of pentagon and one kind of hexagon. While WP is based on such structure, the 4 counter-examples presented in this paper aren't.

2.1. Degree of Compliance with Plateau's Rules

The author newly looked into the aforementioned examination by Sullivan in terms of the compliance with Plateau's rules. As mentioned in Sec. 1, they represent the following two requirements.

- Surfaces of soap bubbles always meet in threes along an edge called a Plateau border and they do so at an angle of $\cos^{-1}(-1/2)$, which is equal to $120°$.
- These Plateau borders meet in fours at a vertex and they do so at an angle of $\cos^{-1}(-1/3)$, which is equal to approx. $109.471°$, the tetrahedral angle.

In the case of space partitioning polyhedra, the 1st requirement signifies that their dihedrals are equal to $\cos^{-1}(-1/2)$ and the 2nd requirement means that their internal angles are equivalent to $\cos^{-1}(-1/3)$. The rhombic dodecahedron can totally meet the 1st requirement as its dihedrals are all equal to $\cos^{-1}(-1/2)$. While its obtuse angle is equivalent to $\cos^{-1}(-1/3)$, its acute angle is equal to $\cos^{-1}(1/3)$, which is equal to approx. $70.529°$. The latter is smaller than $109.471°$ by about $39°$, which makes its overall compliance with Plateau's law considerably lowered. In the case of the truncated octahedron, on the other hand, there exist two kinds of dihedral. One is equal to $\cos^{-1}(-1/3)$, and the other is equivalent to $\cos^{-1}(-1/\sqrt{3})$. There also exist two kinds of internal angle. One is the right angle $(= 90°)$ and the other is equal to $\cos^{-1}(-1/2)$. Neither of them follows Plateau's rules. The reasons are that all faces of the rhombic dodecahedron are tetragons and that 6 faces out of 14 of the truncated octahedron are also tetragons. The average internal angle of the tetragon is equal to $90°$ and, therefore, it is less than $\cos^{-1}(-1/3)$ by almost $20°$.

If the polyhedra consist of pentagons, a difference between their internal angles and $\cos^{-1}(-1/3)$ is significantly small because their average internal angle is equal to $108°$. Furthermore, if the number of pentagonal faces is 12, a difference between their dihedrals and $\cos^{-1}(-1/2)$ is also pretty small because their average dihedral is about $116.5°$. Unfortunately, however, pentagonal dodecahedra alone cannot partition space with no clearance gap left. For this reason, hexagonal faces are needed. Its quantity, however, should be requisite minimum as its average internal angle is $\cos^{-1}(-1/2)$, which is larger than $\cos^{-1}(-1/3)$ by approx. $10°$. Consideration in terms of symmetry indicates that the minimum required quantity of hexagon is two. Furthermore, the number of faces of polyhedra containing hexagons will inevitably be 14 because it is better for their dihedrals to be close to $\cos^{-1}(-1/2)$ as much as possible. Namely, combinations of a pentagonal dodecahedron and a polyhedron with 14 faces like Goldberg tetrakaidecahedron are considered most advisable in order to raise the degree of compliance with Plateau's rules.

2.2. Six Salient Features

How these polyhedra are assembled in a way to eliminate interspace is shown in Fig.2. The top view as well as the side view of the Goldberg tetrakaidecahedron are illustrated upper left and center right, respectively. Particularly, the top view represents the tetrakaidecahedron projected to an X-Y plane with the Z-axis serving as 2-fold rotational symmetry axis. The hexagon is on a plane parallel to the projection plane and, therefore, the length of all its edges represents its actual length. The same can be said of the base of the larger pentagon and its diagonal parallel to the base. Shown lower left, on the other hand, is the pentagonal dodecahedron projected to an X-Y plane with the Z-axis serving as 2-fold rotational symmetry axis. In Fig.2, $\overrightarrow{BE}$ and $\overrightarrow{BH}$ both represent the longest diagonal of the hexagon and $\overrightarrow{DO_3}$ is the same in length as $\overrightarrow{FO_5}$.

The partitions based on the combinations of such dodecahedra and tetrakaidecahedra have 6 distinctive features as mentioned below.

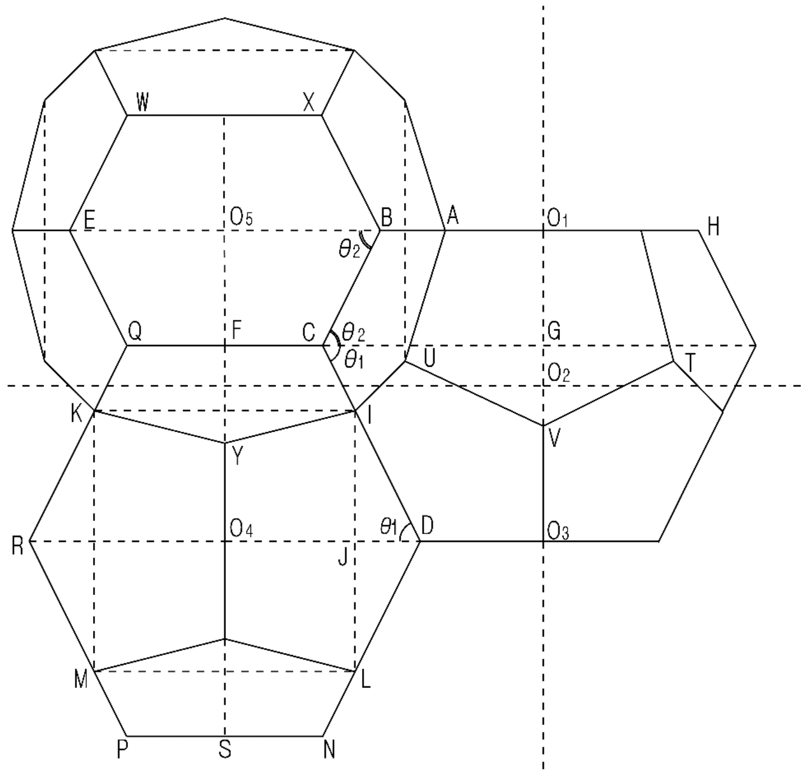

Fig.2 Combination of pentagonal dodecahedra and GB tetrakaidecahedra

<u>Feature 1</u>: The length of the longest diagonal of the hexagon $\overrightarrow{BE}\ (=\overrightarrow{BH})$ is equal to the height of the Goldberg tetrakaidecahedron $\overrightarrow{O_1O_3}$ connecting the center of the top hexagon O_1 to the center of the bottom hexagon O_3.

[Proof] Here $\angle DCG$ in the side view of the tetrakaidecahedron is named θ_1 and $\angle BCG$ in the side view of the tetrakaidecahedron is named θ_2. Then,

$$\overrightarrow{BO_1} = \overrightarrow{BO_5} = \overrightarrow{CF} + \overrightarrow{BC}\cdot\cos\theta_2 \tag{2.2.1}$$

$$\overrightarrow{O_1O_2} = \frac{\overrightarrow{O_1O_3}}{2} = \left(\overrightarrow{CD}\cdot\sin\theta_1 + \overrightarrow{BC}\cdot\sin\theta_2\right)/2 \tag{2.2.2}$$

$$\overrightarrow{BO_1} + \overrightarrow{BC}\cdot\cos\theta_2 = \overrightarrow{DO_3} + \overrightarrow{CD}\cdot\cos\theta_1 \tag{2.2.3}$$

$$\overrightarrow{DO_3} = \overrightarrow{FO_5} = \overrightarrow{BC}\cdot\sin\theta_2.$$

Therefore, the following formula holds true:

$$\overrightarrow{BO_1} + \overrightarrow{BC}\cdot\cos\theta_2 = \overrightarrow{BC}\cdot\sin\theta_2 + \overrightarrow{CD}\cdot\cos\theta_1 \tag{2.2.4}$$

The following formula also holds true in the side elevation of the dodecahedron (lower left).

$$\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{CD}} \cdot \cos\theta_1 = \overrightarrow{\mathrm{CD}} \cdot \sin\theta_1 \qquad (2.2.5)$$

Here, at this point, we substitute $\overrightarrow{\mathrm{CD}} \cdot \sin\theta_1$ in the formula (2.2.2) with the one in the formula (2.2.5). Then, we get the following formula:

$$\overrightarrow{\mathrm{O_1O_2}} = \left(\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{CD}} \cdot \cos\theta_1 + \overrightarrow{\mathrm{BC}} \cdot \sin\theta_2\right)/2 \qquad (2.2.6)$$

Here, at this point, we rewrite the formula (2.2.3) as follows:

$$\overrightarrow{\mathrm{CD}} \cdot \cos\theta_1 = \overrightarrow{\mathrm{BO_1}} + \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2 - \overrightarrow{\mathrm{BC}} \cdot \sin\theta_2 \qquad (2.2.7)$$

Here, at this point, we substitute $\overrightarrow{\mathrm{CD}} \cdot \cos\theta_1$ in the formula (2.2.6) with the one in the formula (2.2.7). Then, we get the following formula:

$$\begin{aligned}\overrightarrow{\mathrm{O_1O_2}} &= \left(\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{BO_1}} + \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2 - \overrightarrow{\mathrm{BC}} \cdot \sin\theta_2 + \overrightarrow{\mathrm{BC}} \cdot \sin\theta_2\right)/2 \\ &= \left(\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{BO_1}} + \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2\right)/2 \\ &= \overrightarrow{\mathrm{BO_1}}/2 + \left(\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2\right)/2 \end{aligned} \qquad (2.2.8)$$

Here, at this point, we substitute $\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2$ in the formula (2.2.8) with the one in the formula (2.2.1). Then, we get the following formula:

$$\overrightarrow{\mathrm{O_1O_2}} = \mathrm{BO_1}/2 + \overrightarrow{\mathrm{BO_1}}/2 = \overrightarrow{\mathrm{BO_1}} \qquad (2.2.9)$$

Therefore, $\overrightarrow{\mathrm{O_1O_3}} = 2 \cdot \overrightarrow{\mathrm{O_1O_2}} = 2 \cdot \overrightarrow{\mathrm{BO_1}} = \overrightarrow{\mathrm{BH}} \left(= \overrightarrow{\mathrm{BE}}\right)$

Feature 2: The angle CBE (∠CBE) in the top view of the tetrakaidecahedron is equal to ∠BCG in the side view of the tetrakaidecahedron. This is self-explanatory as $\overrightarrow{\mathrm{BE}}$ and $\overrightarrow{\mathrm{CG}}$ are parallel to each other.

Feature 3: The angle $\mathrm{BO_1O_2}$ ($\angle \mathrm{BO_1O_2}$) is 90°. $\overrightarrow{\mathrm{BO_2}}$ is a space diagonal connecting the vertex B with the median point $\mathrm{O_2}$ of the tetrakaidecahedron. Coupled with these facts, the formula (2.2.9) above indicates that the length ratio of $\overrightarrow{\mathrm{BO_1}}$ to $\overrightarrow{\mathrm{BO_2}}$ is $1{:}\sqrt{2}$, which signifies that the length ratio of $\overrightarrow{\mathrm{BO_2}}$ to $\overrightarrow{\mathrm{BH}}$ is $1{:}\sqrt{2}$, too.

Feature 4: A dihedral is made up by 2 smaller pentagons of the Goldberg tetrakaidecahedron, sharing their shortest edge (base). There exist 4 lots of such pentagons. One of these lots is composed of pentagons ABCIU and $\mathrm{VO_3DIU}$. A dihedral is made up with $\overrightarrow{\mathrm{IU}}$ serving as the base shared. Space-filing is achieved with 3 of this dihedral getting together. It means that this dihedral always is equal to $\cos^{-1}(-1/2)$.

Feature 5: In Fig.2, the edges $\overrightarrow{\mathrm{KI}}$, $\overrightarrow{\mathrm{IL}}$, $\overrightarrow{\mathrm{LM}}$, $\overrightarrow{\mathrm{MK}}$ of the pentagonal dodecahedron are all parallel to the projection plane and, therefore, they represent their actual length. They are all diagonals of the larger pentagon parallel to the base and, accordingly, they are equal in length. This means that the tetragon KILM is a square, which indicates that the length ratio of its edge (e.g. $\overrightarrow{KI}$) to its diagonal (e.g. $\overrightarrow{KL}$) is 1:$\sqrt{2}$. Whereas such edge of the square is a diagonal of the larger pentagon parallel to the base, the diagonal of the square is a space diagonal of the dodecahedron.

<u>Feature 6</u>: In the pentagonal dodecahedron, there exist six edges serving as the base of the pentagon. If their respective centers are mutually connected, a regular octahedron is created, which contains three squares crossing each other at right angles at their median point. Refer to Fig.3. If two of them that are symmetrical with respect to the median point are selected, a vector connecting them serves as a diagonal of such square and vectors connecting them with the other centers serve as its edges. As a matter of course, the length ratio of the edge to the diagonal is the square root of 2 ratio (1:$\sqrt{2}$).

Fig.2 also indicates that $\angle \mathrm{DCG}\ (= \theta_1)$ is equal to $\angle \mathrm{CDO_4}$. This angle is equal to one half of the dihedral of the dodecahedron made up by the longer edge of the pentagon.

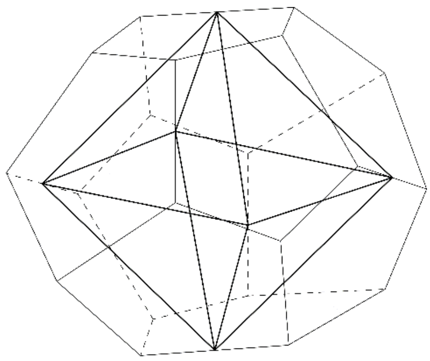

Fig.3 Regular octahedron contained in the pentagonal dodecahedron

In view of the foregoing, the following are conjectured to be two key requirements for the most efficient partition of space with minimal surface area.

- <u>Square root of 2 ratio</u>: WP contain this ratio three-dimensionally. The Rhombic dodecahedron and the truncated octahedron meet this requirement two-dimensionally as well as three-dimensionally.
- <u>Plateau's rules</u>: WP partially complies with these rules. The rhombic dodecahedron follows these rules to a large extent. The reason is that its dihedrals are all equal to $\cos^{-1}(-1/2)$ and that its obtuse angle is equivalent to $\cos^{-1}(-1/3)$. In the case of the truncated octahedron, however, the degree of compliance to these rules is not so high.

3. How to Newly Design Pentagonal Dodecahedron and Goldberg Tetrakaidecahedron

The above-mentioned 4 counter-examples to Kelvin's conjecture have all been newly designed. The method to make it happen is explained in detail below

In Fig.2 lower left, the pentagonal dodecahedron is projected to an X-Y plane with the Z-axis serving as 2-fold rotational symmetry axis. The segment CD and DN represent a larger pentagon of the Goldberg tetrakaidecahedron and a pentagon of the dodecahedron. These pentagons are congruent and, therefore, precisely lie on top of each other. Since these pentagons are perpendicular to the projection plane, $\overrightarrow{\mathrm{CD}}$ and $\overrightarrow{\mathrm{DN}}$ represent a perpendicular drawn from an apex of the pentagon to its base. They indicate their actual length because they are parallel to the projection plane. For the same reason, $\overrightarrow{\mathrm{CQ}}$, $\overrightarrow{\mathrm{NP}}$, $\overrightarrow{\mathrm{IK}}$, $\overrightarrow{\mathrm{KM}}$, $\overrightarrow{\mathrm{LM}}$, $\overrightarrow{\mathrm{IL}}$ all indicate their respective actual length. In Fig.2 lower left, the length of $\overrightarrow{\mathrm{FS}}$ and $\overrightarrow{\mathrm{DR}}$ is the same and, understandably, the length of $\overrightarrow{\mathrm{FO_4}}$ and $\overrightarrow{\mathrm{DO_4}}$ is also the same, where $\mathrm{O_4}$ represents the median point of the dodecahedron. In this figure, the tetragon IKML represents a square and, therefore, $\overrightarrow{\mathrm{IJ}}$ and $\overrightarrow{\mathrm{JO_4}}$ are equal in length. Here, the following equations can be written:

$$\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{CD}} \cdot \cos\theta_1 = \overrightarrow{\mathrm{CD}} \cdot \sin\theta_1$$

$$\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{CI}} \cdot \cos\theta_1 = \left(\overrightarrow{\mathrm{CD}} - \overrightarrow{\mathrm{CI}}\right) \cdot \sin\theta_1$$

$$\sin\theta_1 = \sqrt{1 - (\cos\theta_1)^2}$$

Fig.2 includes a side view of the Goldberg tetrakaidecahedron. The segment BC represents a 2nd longer edge of smaller pentagon, where B is its apex. Since $\overrightarrow{\mathrm{BC}}$ is parallel to the projection plane, it indicates the actual length. The angle BCG (∠BCG) which is equal to ∠$\mathrm{CBO_5}$ and ∠$\mathrm{CDO_4}$ (=∠DCG) are called θ_2 and θ_1, respectively. Such naming was already done in Sec. 2.2. Here, $\overrightarrow{\mathrm{CI}}, \overrightarrow{\mathrm{CD}}, \sin\theta_1, \cos\theta_1, \overrightarrow{\mathrm{BC}}, \sin\theta_2, \cos\theta_2$ and $\overrightarrow{\mathrm{CF}}$ are respectively substituted by variables x_1, x_2, x_3, x_4, x_5, x_6, x_7 and x_8. Namely,

$$x_1 = \overrightarrow{\mathrm{CI}},\ x_2 = \overrightarrow{\mathrm{CD}},\ x_3 = \sin\theta_1,\ x_4 = \cos\theta_1,$$
$$x_5 = \overrightarrow{\mathrm{BC}},\ x_6 = \sin\theta_2,\ x_7 = \cos\theta_2,\ x_8 = \overrightarrow{\mathrm{CF}}$$

Then, the aforementioned three equations are rewritten as follows:

$$x_8 + x_2 \cdot x_4 = x_2 \cdot x_3 \tag{3.1}$$

$$x_8 + x_1 \cdot x_4 = (x_2 - x_1) \cdot x_3 \tag{3.2}$$

$$x_3 = \sqrt{1 - {x_4}^2} \tag{3.3}$$

Further, using these variables, coordinates of each vertex of the Goldberg tetrakaidecahedron are expressed as follows, with the coordinate origin being $\mathrm{O_2}$.

- x–coordinate of $\mathrm{B} = -\left(\overrightarrow{\mathrm{AO_1}} + \overrightarrow{\mathrm{AB}}\right) = -\left(\overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2\right) = -(x_8 + x_5 \cdot x_7)$
- x–coordinate of $\mathrm{C} =$ x–coordinate of $\mathrm{B} - \overrightarrow{\mathrm{BC}} \cdot \cos\theta_2$
  $= -(x_8 + x_5 \cdot x_7) - x_5 \cdot x_7 = -(x_8 + 2 \cdot x_5 \cdot x_7)$
- z–coordinate of $\mathrm{A, B, H, O_1} = \left(\overrightarrow{\mathrm{CD}} \cdot \sin\theta_1 + \overrightarrow{\mathrm{BC}} \cdot \sin\theta_2\right)/2 = (x_2 \cdot x_3 + x_5 \cdot x_6)/2$
- x–coordinate of $\mathrm{T} = \overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{CI}} \cdot \cos\theta_1 = x_8 + x_1 \cdot x_4$
- y–coordinate of $\mathrm{T, U} =$ y–coordinate of $\mathrm{A} - \left(\overrightarrow{\mathrm{CD}} - \overrightarrow{\mathrm{CI}}\right) \cdot \cos\theta_1$
  $= -x_5 \cdot x_6 - (x_2 - x_1) \cdot x_4$
- z–coordinate of $\mathrm{T, U} = -($z–coordinate of I$)$
  $=$ z–coordinate of $\mathrm{A, B, H, O_1} - \left(\overrightarrow{\mathrm{CD}} - \overrightarrow{\mathrm{CI}}\right) \cdot \sin\theta_1$

$$= \frac{(x_2 \cdot x_3 + x_5 \cdot x_6)}{2} - (x_2 - x_1) \cdot x_3 = (x_5 \cdot x_6 + 2 \cdot x_1 \cdot x_3 - x_2 \cdot x_3)/2$$

- y−coordinate of V = x−coordinate of C = $-(x_8 + 2 \cdot x_5 \cdot x_7)$
- z−coordinate of C = z−coordinate of A, B, H, $O_1 - \overrightarrow{BC} \cdot \sin\theta_2$
$$= -\left\{\frac{(x_2 \cdot x_3 + x_5 \cdot x_6)}{2} - x_5 \cdot x_6\right\} = -\frac{(x_5 \cdot x_6 - x_2 \cdot x_3)}{2} = \frac{(x_2 \cdot x_3 - x_5 \cdot x_6)}{2}$$
- z−coordinate of V = −(z−coordinate of C)
$$= -\frac{(x_2 \cdot x_3 - x_5 \cdot x_6)}{2} = (x_5 \cdot x_6 - x_2 \cdot x_3)/2$$
- x−coordinate of I = x−coordinate of C + $\overrightarrow{CI} \cdot \cos\theta_1 = -(x_8 + 2 \cdot x_5 \cdot x_7) + x_1$
- y−coordinate of I = −(x−coordinate of T) = $-(x_8 + x_1 \cdot x_4)$
- z−coordinate of I = −(z−coordinate of T, U)
$$= -(x_5 \cdot x_6 + 2 \cdot x_1 \cdot x_3 - x_2 \cdot x_3)/2$$
- x−coordinate of D = y−coordinate of A = $-x_5 \cdot x_6$
- y−coordinate of D = $-\overrightarrow{CF} = -x_8$
- z−coordinate of D, O_3 = −(z−coordinate of A, B, H, O_1) = $-(x_2 \cdot x_3 + x_5 \cdot x_6)/2$

The side view of the Goldberg tetrakaidecahedron indicates that the following equation holds true.

$$\overrightarrow{BO_1} + \overrightarrow{BC} \cdot \cos\theta_2 = \overrightarrow{CG} = \overrightarrow{DO_3} + \overrightarrow{CD} \cdot \cos\theta_1$$

The length of vectors $\overrightarrow{BO_1}$ and $\overrightarrow{DO_3}$ are represented by x−coordinate of B and x−coordinate of D, respectively. Therefore, this equation is rewritten as follows.

$$x_8 + 2x_5 \cdot x_7 - (x_5 \cdot x_6 + x_2 \cdot x_4) = 0 \quad (3.4)$$

Based on the aforementioned formula (2.2.9), the following formula is realized.

$$-(x\text{−coordinate of B}) = \text{z−coordinate of A, B, H, } O_1.$$

Therefore, the following equation holds true.

$$x_8 + x_5 \cdot x_7 - (x_2 \cdot x_3 + x_5 \cdot x_6)/2 = 0 \quad (3.5)$$

In Fig.2,

$$\overrightarrow{BO_5} = \overrightarrow{BO_1} = \overrightarrow{O_1O_2} = \overrightarrow{O_2O_3}$$

Therefore, the following formula is realized.

$$\overrightarrow{O_1O_5} = \overrightarrow{O_4O_5}$$

Namely,

$$2 \cdot \left(\overrightarrow{FC} + \overrightarrow{BC} \cdot \cos\theta_2\right) = \overrightarrow{BC} \cdot \sin\theta_2 + \overrightarrow{CD} \cdot \sin\theta_1$$

With the aforementioned variables, this equation can be rewritten as follows.

$$2 \cdot (x_8 + x_5 \cdot x_7) = x_5 \cdot x_6 + x_2 \cdot x_3 \quad (3.6)$$

The volume of the pentagonal dodecahedron V_{12} must be equal to the one of the Goldberg tetrakaidecahedron V_{14}. Therefore, the following equation holds true.

$$V_{12} - V_{14} = 0 \quad (3.7)$$

The larger pentagons are all bilaterally symmetric and, accordingly, they are divided into triangles symmetrically. In case of the smaller pentagon, the area of the right triangle S_{spr} and the one of the left triangle S_{spl} are the same. For this reason, the following equation holds true.

$$S_{spr} - S_{spl} = 0 \quad (3.8)$$

Based on the theorem of the trigonometrical function, $\sin\theta_2 = \sqrt{1-(\cos\theta_2)^2}$. Therefore, the following equation holds true.

$$x_6 = \sqrt{1-(x_7)^2} \quad (3.9)$$

4. Four New Counter-examples

WP is one of infinite kinds of partition based on the tetrahedrally close-packed structures and is currently considered to be the most efficient partition of space with minimal surface area. The author conjectured, as stated at the end of Sec. 3, that there exist two key requirements for such partition: (1) square root of 2 ratio, (2) Plateau's rules. WP meets both of these requirements to a degree.

WP has 7 different kinds of dihedral, i.e. two kinds contained in the pentagonal dodecahedron and five kinds contained in the Goldberg tetrakaidecahedron. It is to be pointed out here that only one kind of dihedral out of such seven is equal to 120°. In the light of this, the isoperimetric quotient c could be further improved if the Plateau's rules are followed to a larger extent.

It is to be noted that WP contains the square root of 2 ratio only three-dimensionally. Accordingly, the isoperimetric quotient is expected to be further enhanced if faces of the polyhedra include the square root of 2 ratio, too. The pentagonal dodecahedron consists of 12 larger pentagons and the Goldberg tetrakaidecahedron is composed of 2 hexagons, 4 larger pentagons, and 8 smaller pentagons. This indicates that the larger pentagon has the largest weight, among other polygons, exerting the biggest influence on the isoperimetric quotient c. For this reason, the author decided to include the square root of 2 ratio in it. The larger pentagon has three kinds of diagonal: ⟨1⟩ the one parallel to the base; ⟨2⟩ the one extending from either end of the base to the end of the diagonal⟨1⟩; ⟨3⟩ the one extending from either end of the base to the apex. Based on such diagonal, the following three kinds of length ratio are created.

1. Length ratio of the base to the diagonal⟨1⟩.
2. Length ratio of the base to the diagonal⟨2⟩.
3. Length ratio of the base to the diagonal⟨3⟩.

The next step is to newly design larger pentagons in a way to have either of such length ratios become the square root of 2 ratio.

Geometrical quantities of a pentagonal dodecahedron and a Goldberg tetrakaidecahedron designed anew are figured out in the following way. There are 8 cells in a cubic $2 \times 2 \times 2$ flat torus, which start as Voronoi cells on centers below.

0	0	0
1	1	1
0.5	0	1
1.5	0	1
0	1	0.5
0	1	1.5
1	0.5	0
1	1.5	0

The volume of the pentagonal dodecahedron V_{12} and the Goldberg tetrakaidecahedron V_{14} is equal to 1. Therefore, the following equation holds true.

$$V_{12} - 1 = 0 \tag{4.0}$$

4.1. New Counter-example No.1 (following the Plateau's rules to a larger extent)

As mentioned earlier, the pentagonal dodecahedron has 2 kinds of dihedral, i.e. the larger dihedral made up by its longer edge and the smaller one made up by its shorter edge. In case of WP, this dihedral is $\cos^{-1}(-3/5)$, which is equal to approx. $126°52'63"$. In an effort to further enhance the isoperimetric quotient, the author has set the larger dihedral at 120° and newly designed the pentagonal dodecahedron and the Goldberg tetrakaidecahedron based on such setting. In this case, $\theta_1 = 60°$ and, therefore, $\sin\theta_1$ and $\cos\theta_1$ of the Eqs. (3.1) through (3.3) are equal to $\sqrt{3}/2$ and $1/2$, respectively. Namely, $x_3 = \sqrt{3}/2$, $x_4 = 1/2$.

The next step is to calculate geometrical quantities of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron. As mentioned in Sec. 2.2, the dihedral φ_1 made up by 2 smaller pentagons of the Goldberg tetrakaidecahedron sharing their shortest edge (base) must be $\cos^{-1}(-1/2)$, which is equal to $120°$. Accordingly, the following equation holds true.

$$\cos\varphi_1 - 0.5 = 0 \tag{4.1}$$

The Eqs. (3.1) through (3.9) and (4.0) are nonlinear simultaneous equations with x_1, x_2, x_5, x_6, x_7, x_8 being variables. Solutions of these equations are as follows.

$$x_1 = \overrightarrow{\mathrm{CI}} \approx 0.2755734$$
$$x_2 = \overrightarrow{\mathrm{CD}} \approx 0.7528806$$
$$x_5 = \overrightarrow{\mathrm{BC}} \approx 0.4140794$$
$$x_6 = \sin\theta_2 \approx 0.8403854$$
$$x_7 = \cos\theta_2 \approx 0.5419892$$
$$x_8 = \overrightarrow{\mathrm{CF}} \approx 0.2755734$$

This means that the length of $\overrightarrow{\mathrm{CF}}$ and $\overrightarrow{\mathrm{CI}}$ is the same. Shown in Fig.4 is the larger pentagon making up the pentagonal dodecahedron and the Goldberg tetrakaidecahedron, in which the angle $\mathrm{ICI_1}$ is named σ_1. (Note: CE stands for counter-example.)
In this figure, the following equations hold true.

$$\overrightarrow{\mathrm{CI_1}}^2 = \overrightarrow{\mathrm{CI}}^2 + \overrightarrow{\mathrm{II_1}}^2$$

$$\sigma_1 = \tan^{-1}\left(\overrightarrow{\mathrm{II_1}}/\overrightarrow{\mathrm{CI}}\right)$$

Here, the length of $\overrightarrow{\mathrm{II_1}}$ is the same as the aforementioned x-coordinate of T. Therefore,

$$\overrightarrow{\mathrm{II_1}} = \overrightarrow{\mathrm{CF}} + \overrightarrow{\mathrm{CI}} \cdot \cos\theta_1 \approx 0.2755734 + 0.2755734 \cdot 1/2 \approx 0.4133601$$

$$\overrightarrow{\mathrm{CI_1}} \approx \sqrt{(0.2755734)^2 + (0.4133601)^2} \approx 0.4967970$$

$$\sigma_1 \approx 56.30993° \approx 56°18'35.76"$$

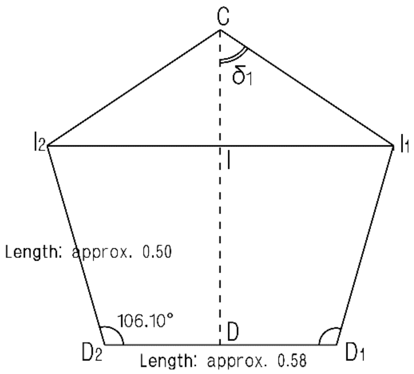

Fig.4 Larger pentagon of the CE No.1

Thus, the shape of the larger pentagon and the pentagonal dodecahedron is fixed. Further, the valuable θ_2 was calculated from the values of $\sin\theta_2$ and $\cos\theta_2$.

$$\theta_2 \approx 57.18084° \approx 57°10'51.03"$$

The coordinates of five vertices of the smaller pentagon indicate that they are not on the same plane. The calculation of the volume and the total surface area of the Goldberg tetrakaidecahedron is based on the partition of smaller pentagons into three triangles, i.e. two congruent triangles right and left and the isosceles triangle in the center as depicted in Fig.5.

The base of the former also serves as the oblique side of the latter and as the diagonal of the pentagon. The left-hand diagonal and the right-hand diagonal respectively make up a dihedral. The dihedral created by the former stands at approx. $171.3974°(\approx 171°23'50.85")$ and the one created by the latter stands at approx. $188.6025°(\approx 188°36'9.15")$. The former is about $8°36'9.15"$ smaller than the plane angle and the latter is about $8°36'9.15"$ larger than that. In other words, these dihedrals are the same. The former is mountain-folded and the latter is valley-folded.

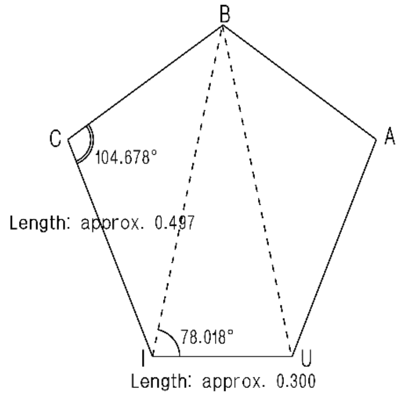

Fig.5 Smaller pentagon of the CE No. 1

With all these points taken into account, it has become clear that the polygon having initially been regarded as smaller pentagon is actually composed of three triangles and that two of them achieve space-filling locally when they are superposed with each other. This means that the polyhedron initially considered as tetrakaidecahedron is actually a triacontahedron.

With all these geometrical quantities, all the coordinates of the pentagonal dodecahedron and the triacontahedron can be figured out.

The volume V, total surface area A, A^3/V^2, $c\,(=A/V^{2/3})$ of the pentagonal dodecahedron and the triacontahedron have been calculated. The average of A^3/V^2 and $c\,(=A/V^{2/3})$ of these polyhedra based on the ratio of 1:3 have also been figured out. They are summarized in Table 1 below.

The isoperimetric quotient c of the new counter-example No.1 thus calculated is marginally better than the flattened WP structure mentioned in the Abstract. The latter stands at approx. 5.296998. This flattened structure is hereinafter called fWP to be distinguished from WP.

Pentagonal dodecahedron	Volume V	1.000000
	Total surface area A	5.312928
	A^3/V^2	149.9691
	Isoperimetric quotient $c\,(=A/V^{2/3})$	5.312928
Triaconta-hedron	Volume V	1.000000
	Total surface area A	5.291425
	A^3/V^2	148.1556
	Isoperimetric quotient $c\,(=A/V^{2/3})$	5.291425
Average at 1 : 3	A^3/V^2	148.6089
	isoperimetric quotient $c\,(=A/V^{2/3})$	5.296817

Table 1 Isoperimetric quotient and the other key geometric quantities of CE No.1

4.2. New Counter-example No.2

The next step taken by the author is to have the larger pentagon newly designed under the condition that the length ratio of the base to the diagonal⟨1⟩ is $1{:}\sqrt{2}$. The diagonal⟨1⟩ here is the one parallel to the base. Refer to Fig.6 below.

Geometrical quantities of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron were calculated based on the same approach as the aforementioned new counter-example No.1.

$\overrightarrow{\mathrm{CI}} \approx 0.2561799$
$\overrightarrow{\mathrm{CD}} \approx 0.7295387$
$\overrightarrow{\mathrm{CF}} \approx 0.2943704$
$\theta_1 \approx 61.57792° \ (\approx 61°34'40.54")$
$\overrightarrow{\mathrm{II_1}} \approx 0.4163026$
$\overrightarrow{\mathrm{CI_1}} \approx 0.4888108$
$\sigma_1 \approx 58.39306° \ (\approx 58°23'35.25")$
$\overrightarrow{\mathrm{BC}} \approx 0.4131964$
$\theta_2 \approx 60.15498° \ (\approx 60°09'17.88")$

With these values, the shape of the larger pentagon and the pentagonal dodecahedron is fixed. Fig.6 below illustrates this pentagon. The base and the diagonal parallel to it are indicated in bold solid line.

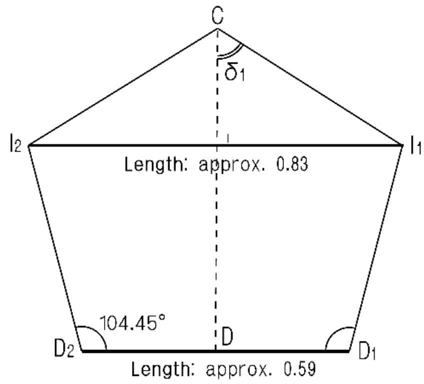

Fig.6 Larger pentagon of the CE No.2

As in the case of the new counter-example No.1, five vertices of the smaller pentagon are not on the same plane. Refer to Fig.7 below.

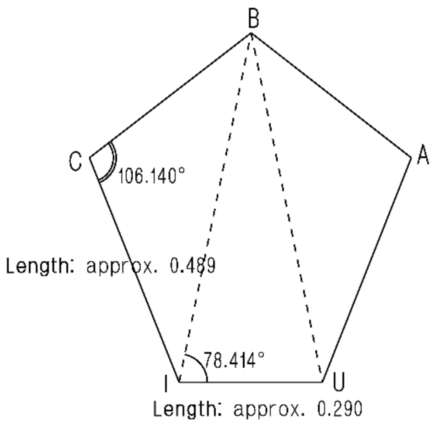

Fig.7 Smaller pentagon of the CE No.2

The use of the same approach revealed that the dihedrals created by the left-hand diagonal and the right-hand diagonal stand at approximately $175.3979°(\approx 175°23'52.53")$ and approximately $184.6020°(\approx 184°36'07.47")$, respectively. These dihedrals are the same and their difference from the plane angle is about $4°36'07.47"$. The former is mountain-folded and the latter is valley-folded. As in the case of the new counter-example No.1, the polyhedron initially considered as tetrakaidecahedron actually turned out to be a triacontahedron.

With all these geometrical quantities, all the coordinates of the vertices of the pentagonal dodecahedron and the triacontahedron can be figured out.

The volume V, total surface area A, A^3/V^2 and $c\ (= A/V^{2/3})$ of the pentagonal dodecahedron and the triacontahedron have been calculated. The average of A^3/V^2 and $c\ (= A/V^{2/3})$ of these polyhedral based on the ratio of 1:3 have also been figured out. They are summarized in Table 2. The space division structure based on the new counter-example No.2 is depicted in Fig.8.

Fig.8 Space division structure based on the CE No. 2

Pentagonal dodecahedron	Volume V	1.000000
	Total surface area A	5.316620
	A^3/V^2	150.2819
	Isoperimetric quotient $c\ (= A/V^{2/3})$	5.316620
Triaconta-hedron	Volume V	1.000000
	Total surface area A	5.284268
	A^3/V^2	147.5552
	Isoperimetric quotient $c\ (= A/V^{2/3})$	5.284268
Average at 1 : 3	A^3/V^2	148.2369
	Isoperimetric quotient $c\ (= A/V^{2/3})$	5.292393

Table 2 Isoperimetric quotient and the other key geometric quantities of CE No.2

Out of 30 faces of the triacontahedron, 24 faces are triangles. They are painted in various different colors. Its isoperimetric quotient c thus calculated is also slightly more superior to the one of fWP (≈ 5.296998).

4.3.   New Counter-example No.3

The next step is to have the larger pentagon newly designed under the condition that the length ratio of the base to the diagonal⟨2⟩ is $1{:}\sqrt{2}$. The diagonal⟨2⟩ here is the one extending from either end of the base to the end of the diagonal parallel to the base.

Geometrical quantities of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron were figured out based on the same approach. The values thus calculated are as follows.

$\overrightarrow{\mathrm{CI}} \approx 0.2448942$
$\overrightarrow{\mathrm{CD}} \approx 0.7161651$
$\overrightarrow{\mathrm{CF}} \approx 0.3052577$
$\theta_1 \approx 62.54150° \ (\approx 62°32'29.44")$
$\overrightarrow{\mathrm{II_1}} \approx 0.4181799$
$\overrightarrow{\mathrm{CI_1}} \approx 0.5336100$
$\sigma_1 \approx 51.59882° \ (\approx 51°35'55.76')$
$\overrightarrow{\mathrm{BC}} \approx 0.4132739$
$\theta_2 \approx 61.88658° \ (\approx 61°53'11.69")$

With these values, the shape of the larger pentagon and the pentagonal dodecahedron is fixed. Refer to Fig.9 below. The aforementioned diagonal and the base are shown in bold solid line.

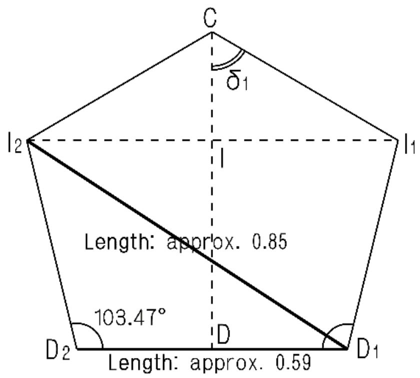

Fig.9 Larger pentagon of the CE No.3

With all these geometrical quantities, all the coordinates of the vertices of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron can be figured out. Like the new counter-example No.1 and No.2, five vertices of the smaller pentagon are not on the same plane. Refer to Fig.10 below.

The use of the same approach revealed that the dihedrals created by the left-hand diagonal and the right-hand diagonal stand at approx. $172.6741° (\approx 172°40'26.60")$ and approx. $187.3259° (\approx 187°19'33.40")$, respectively. These dihedrals are the same and their difference from the plane angle is about $7°19'33.40"$. The former is mountain-folded and the latter is valley-folded. As in the case of the new counter-examples No.1 and No.2, the polyhedron initially considered to be tetrakaidecahedron actually turned out to be a triacontahedron.

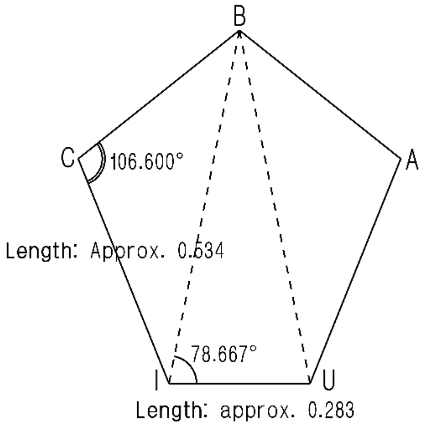

Fig.10 Smaller pentagon of the CE No.3

The volume V, total surface area A, A^3/V^2 and $c\ (= A/V^{2/3})$ of the pentagonal dodecahedron and the triacontahedron have been calculated. The average of A^3/V^2 and $c\ (= A/V^{2/3})$ of these polyhedra based on the ratio of 1:3 have also been figured out. They are summarized in Table 3 below.

Pentagonal dodecahedron	Volume V	1.000000
	Total surface area A	5.320139
	A^3/V^2	150.5805
	Isoperimetric quotient $c\ (= A/V^{2/3})$	5.320139
Triaconta-hedron	Volume V	1.000000
	Total surface area A	5.284593
	A^3/V^2	147.5824
	Isoperimetric quotient $c\ (= A/V^{2/3})$	5.284593
Average at 1 : 3	A^3/V^2	148.3319
	Isoperimetric quotient $c\ (= A/V^{2/3})$	5.293524

Table 3 Isoperimetric quotient and the other key geometric quantities of CE No.3

The isoperimetric quotient c of the new counter-example No.3 thus calculated is also marginally more superior to the one of fWP (≈ 5.296998).

4.4. New Counter-example No.4

The next step is to have the larger pentagon newly designed under the condition that the length ratio of the base to the diagonal⟨3⟩ is 1:$\sqrt{2}$. The diagonal⟨3⟩ here is the one extending from either end of the base to the apex. Refer to Fig.11.

Geometrical quantities of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron were figured out based on the same approach. The values thus calculated are as follows.

$$\overrightarrow{\mathrm{CI}} \approx 0.2692813$$
$$\overrightarrow{\mathrm{CD}} \approx 0.7452607$$
$$\overrightarrow{\mathrm{CF}} \approx 0.2816821$$
$$\theta_1 \approx 60.50136° \ (\approx 60°30'03.89")$$

$\overrightarrow{\mathrm{II_1}} \approx 0.4142770$
$\overrightarrow{\mathrm{CI_1}} \approx 0.4941031$
$\sigma_1 \approx 56.97601° (\approx 56°58'33.63')$
$\overrightarrow{\mathrm{BC}} \approx 0.4136533$
$\theta_2 \approx 58.144426° (\approx 58°08'39.93")$

With these values, the shape of the larger pentagon and the pentagonal dodecahedron is fixed. Refer to Fig.11 below. The aforementioned diagonal and the base are shown in bold solid line.

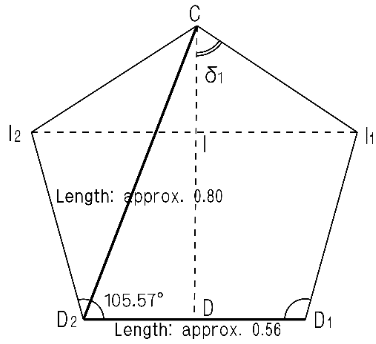

Fig.11 Larger pentagon of the CE No.4

With all these geometrical quantities, all the coordinates of the vertices of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron can be figured out. As in the case of the new counter-example No.1, No.2, and No.3, five vertices of the smaller pentagon are not on the same plane. Refer to Fig.12.

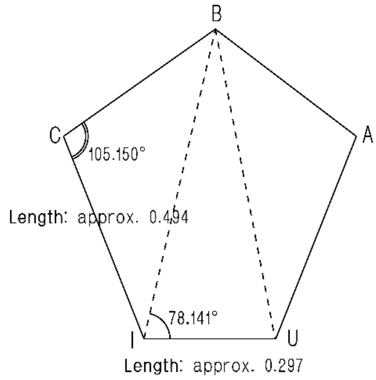

Fig.12 Smaller pentagon of the CE No.4

The use of the same approach revealed that the dihedrals created by the left-hand diagonal and the right-hand diagonal stand at approx. $177.8048°(\approx 177°48'17.26")$ and approx. $182.1952°(\approx 182°11'42.74")$, respectively. These dihedrals are the same and their difference from the plane angle is about $2°11'42.74"$. The former is mountain-folded and the latter is valley-folded. As in the case of the new counter-examples No.1, No.2 and No.3, the polyhedron initially considered to be tetrakaidecahedron actually turned out to be a triacontahedron.

The volume V, total surface area A, A^3/V^2, and $c\ (= A/V^{2/3})$ of the pentagonal dodecahedron and the triacontahedron have been calculated. The average of A^3/V^2 and $c\ (= A/V^{2/3})$ of these polyhedra based on the ratio of 1:3 have also been figured out.

They are summarized in Table 4 below.

Pentagonal dodecahedron	Volume V	1.000000
	Total surface area A	5.313831
	A^3/V^2	150.0456
	Isoperimetric quotient c $(= A/V^{2/3})$	5.313831
Triaconta-hedron	Volume V	1.000000
	Total surface area A	5.288069
	A^3/V^2	147.8738
	Isoperimetric quotient c $(= A/V^{2/3})$	5.288069
Average at 1 : 3	A^3/V^2	148.4168
	Isoperimetric quotient c $(= A/V^{2/3})$	5.294533

Table 4 Isoperimetric quotient and the other key geometric quantities of CE No.4

The isoperimetric quotient c of the new counter-example No.4 thus calculated is also marginally more superior to the one of fWP (≈ 5.296998).

5. Further Reduction of Total Surface Area of the 4 Counter-examples

As mentioned in the Abstract above, space partitioning structure of the 4 counter-examples presented in this paper is all based on the combination of the pentagonal dodecahedron and the triacontahedron. This triacontahedron is similar to the Goldberg tetrakaidecahedron. In the case of the former, 8 smaller pentagons which are included in the latter are respectively composed of 3 triangles. This constitutes a distinctive difference between the two. All faces of these 4 counter-examples are flat and they contain no minimum curved surfaces. Accordingly, these 4 counter-examples were compared with WP with its faces having been flattened out for planar approximation. It is intended for assuring equivalent comparison.

The next step is to further reduce the total surface area of the 4 counter-examples by having them contain minimum surfaces. As in the case of WP, it was accomplished with the Surface Evolver. There are 8 cells in a cubic $2 \times 2 \times 2$ flat torus, which start as Voronoi cells on centers below.

0	0	0
1	1	1
0.5	0	1
1.5	0	1
0	1	0.5
0	1	1.5
1	0.5	0
1	1.5	0

The pentagonal dodecahedron and the triacontahedron were newly designed to have their respective volume become 1. This was already stated in Sec. 4. Out of such eight, two cells are pentagonal dodecahedra and six cells are triacontahedra. Based on the coordinates of their 46 vertices, a Surface Evolver data file was made out for each of the 4 counter-examples. Positions of these vertices are shown in Fig.13 below.

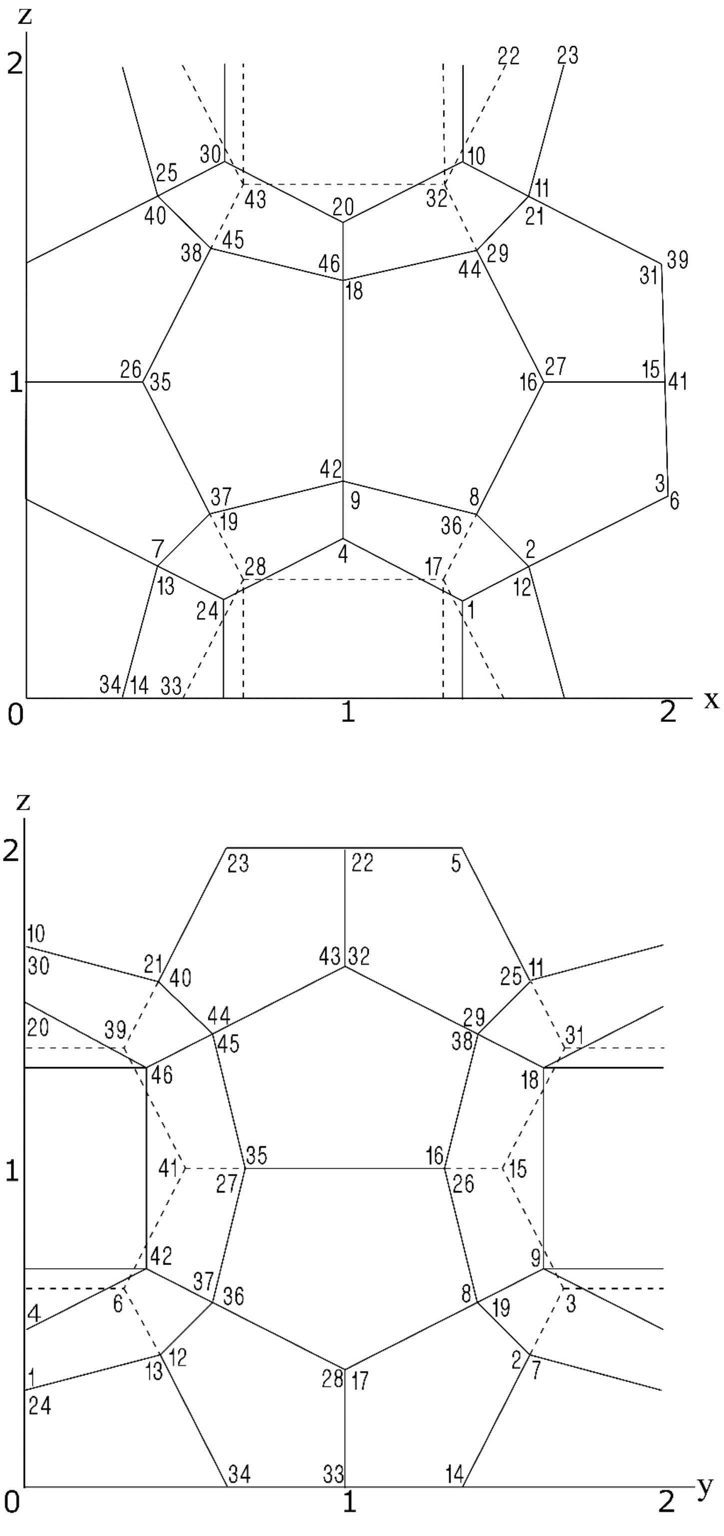

Fig.13 Positions of 46 vertices of the pentagonal dodecahedron and the triacontahedron

Such data file includes 92 edges, 54 faces, and 8 cells along with the coordinates of 46 vertices. Coordinates of 46 vertices of the 4 counter-examples are shown in Appendix 1 and Appendix 2. On the basis of such data file, maximum possible reduction of the total surface area was performed with the Surface Evolver. Results of these calculations are indicated in Table 5 below.

	AREA
Counter-example No.1	21.1539651493413
Counter-example No.2	21.1539649485177
Counter-example No.3	21.1539647904405
Counter-example No.4	21.1539650761630

Table 5 Results of Surface Evolver calculations

The AREA mentioned above represents one half of the total surface area of the eight cells and corresponds to the total surface area of three triacontahedra and one pentagonal dodecahedron.

On the other hand, the AREA of WP was figured out at 21.1539645044697. The coordinates of vertices included in the Surface Evolver data file used for this calculation have seven essential figures. Therefore, the essential figures of the AREA of WP are considered to be seven figures, i.e. 21.15396. For the same reason, comparison between the 4 counter-examples and WP is made with the essential figures of their respective AREA being seven figures, i.e. 21.15396. In other words, these values are virtually the same. Its isoperimetric quotient c is equal to one fourth of the AREA, which is equivalent to 5.288490.

The following firm conclusions can be drawn from the foregoing.

- In the case of space partition based on the combination of the pentagonal dodecahedron and the Goldberg tetrakaidecahedron and on the combination of the pentagonal dodecahedron and the triacontahedron, the most superior isoperimetric quotient is 5.288490. Further enhancement of it is very difficult.
- There exist space partitioning polyhedra other than WP whose isoperimetric quotient is 5.288490. They include the above-mentioned 4 counter-examples.

The space partitioning structure complying with Plateau's rules to the greatest possible extent is considered to be a combination of the pentagonal dodecahedron and the tetrakaidecahedron containing two hexagons. If space partitioning polyhedra different from the aforementioned pentagonal dodecahedron, triacontahedron, or Goldberg tetrakaidecahedron are discovered, the current best isoperimetric quotient 5.288490 may be further improved.

Acknowledgements

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References

[1] T.C. Hales, The Honeycomb Conjecture, Discrete Comput. Geom 25 (2001) 1-22
[2] W. Thompson, Phil. Mag 24, (1887) 503
[3] J. Plateau, Statique Experimental et Theorique des Liquids Soumis aux Seules Forces Moleculaires, Gauthier-Villars
[4] H.M. Princen, P.K. Levinson, J. Colloid Interface Sci. 120 (1987) 172

[5] R. Kusner, J. Sullivan, Forma 11 (1996) 233
[6] D. Weaire, R. Phelan, Philosophical Magazine Letters 69 (1994) 107-110
[7] K.A. Brakke, Exp. Math. 1 (1992) 141
[8] R. Gabbrielli, A new counter-example to Kelvin's conjecture on minimal surfaces, Philosophical Magazine Letters 89 no. 8 (2009) 483-491
[9] J.M. Sullivan, New tetrahedrally close-packed structures, Proceedings of Eurofoam 2000 (Delft) (2000) 111-119

Appendix 1 Coordinates of 46 vertices of Counter-example No.1 and Counter-example No.2

	Counter-example No.1			Counter-example No.2		
	X-coordinates	Y-coordinates	Z-coordinates	X-coordinates	Y-coordinates	Z-coordinates
1	1.3479863	0.0000000	0.2755734	1.3583961	0.0000000	0.2943704
2	1.5866399	1.5866399	0.4133601	1.5836974	1.5836974	0.4163026
3	2.0000000	1.7244266	0.6520137	2.0000000	1.7056296	0.6416039
4	1.0000000	0.0000000	0.5000000	1.0000000	0.0000000	0.5000000
5	1.7244266	1.3479863	2.0000000	1.7056296	1.3583961	2.0000000
6	2.0000000	0.2755734	0.6520137	2.0000000	0.2943704	0.6416039
7	0.4133601	1.5866399	0.4133601	0.4163026	1.5836974	0.4163026
8	1.4133601	1.4133601	0.5866399	1.4163026	1.4163026	0.5836974
9	1.0000000	1.6520137	0.7244266	1.0000000	1.6416039	0.7056296
10	1.3479863	0.0000000	1.7244266	1.3583961	0.0000000	1.7056296
11	1.5866399	1.5866399	1.5866399	1.5836974	1.5836974	1.5836974
12	1.5866399	0.4133601	0.4133601	1.5836974	0.4163026	0.4163026
13	0.4133601	0.4133601	0.4133601	0.4163026	0.4163026	0.4163026
14	0.2755734	1.3479863	0.0000000	0.2943704	1.3583961	0.0000000
15	2.0000000	1.5000000	1.0000000	2.0000000	1.5000000	1.0000000
16	1.6520137	1.2755734	1.0000000	1.6416039	1.2943704	1.0000000
17	1.2755734	1.0000000	0.3479863	1.2943704	1.0000000	0.3583961
18	1.0000000	1.6520137	1.2755734	1.0000000	1.6416039	1.2943704
19	0.5866399	1.4133601	0.5866399	0.5836974	1.4163026	0.5836974
20	1.0000000	0.0000000	1.5000000	1.0000000	0.0000000	1.5000000
21	1.5866399	0.4133601	1.5866399	1.5836974	0.4163026	1.5836974
22	1.5000000	1.0000000	2.0000000	1.5000000	1.0000000	2.0000000
23	1.7244266	0.6520137	2.0000000	1.7056296	0.6416039	2.0000000
24	0.6520137	0.0000000	0.2755734	0.6416039	0.0000000	0.2943704
25	0.4133601	1.5866399	1.5866399	0.4163026	1.5836974	1.5836974
26	0.3479863	1.2755734	1.0000000	0.3583961	1.2943704	1.0000000
27	1.6520137	0.7244266	1.0000000	1.6416039	0.7056296	1.0000000
28	0.7244266	1.0000000	0.3479863	0.7056296	1.0000000	0.3583961
29	1.4133601	1.4133601	1.4133601	1.4163026	1.4163026	1.4163026
30	0.6520137	0.0000000	1.7244266	0.6416039	0.0000000	1.7056296
31	2.0000000	1.7244266	1.3479863	2.0000000	1.7056296	1.3583961
32	1.2755734	1.0000000	1.6520137	1.2943704	1.0000000	1.6416039
33	0.5000000	1.0000000	0.0000000	0.5000000	1.0000000	0.0000000
34	0.2755734	0.6520137	0.0000000	0.2943704	0.6416039	0.0000000
35	0.3479863	0.7244266	1.0000000	0.3583961	0.7056296	1.0000000
36	1.4133601	0.5866399	0.5866399	1.4163026	0.5836974	0.5836974
37	0.5866399	0.5866399	0.5866399	0.5836974	0.5836974	0.5836974
38	0.5866399	1.4133601	1.4133601	0.5836974	1.4163026	1.4163026
39	2.0000000	0.2755734	1.3479863	2.0000000	0.2943704	1.3583961
40	0.4133601	0.4133601	1.5866399	0.4163026	0.4163026	1.5836974
41	2.0000000	0.5000000	1.0000000	2.0000000	0.5000000	1.0000000
42	1.0000000	0.3479863	0.7244266	1.0000000	0.3583961	0.7056296
43	0.7244266	1.0000000	1.6520137	0.7056296	1.0000000	1.6416039
44	1.4133601	0.5866399	1.4133601	1.4163026	0.5836974	1.4163026
45	0.5866399	0.5866399	1.4133601	0.5836974	0.5836974	1.4163026
46	1.0000000	0.3479863	1.2755734	1.0000000	0.3583961	1.2943704

Appendix 2 Coordinates of 46 vertices of Counter-example No.3 and Counter-example No.4

	Counter-example No.3			Counter-example No.4		
	X-coordinates	Y-coordinates	Z-coordinates	X-coordinates	Y-coordinates	Z-coordinates
1	1.3645144	0.0000000	0.3052577	1.3513493	0.0000000	0.2816821
2	1.5818201	1.5818201	0.4181799	1.5857230	1.5857230	0.4142770
3	2.0000000	1.6947423	0.6354856	2.0000000	1.7183179	0.6486507
4	1.0000000	0.0000000	0.5000000	1.0000000	0.0000000	0.5000000
5	1.6947423	1.3645144	2.0000000	1.7183179	1.3513493	2.0000000
6	2.0000000	0.3052577	0.6354856	2.0000000	0.2816821	0.6486507
7	0.4181799	1.5818201	0.4181799	0.4142770	1.5857230	0.4142770
8	1.4181799	1.4181799	0.5818201	1.4142770	1.4142770	0.5857230
9	1.0000000	1.6354856	0.6947423	1.0000000	1.6486507	0.7183179
10	1.3645144	0.0000000	1.6947423	1.3513493	0.0000000	1.7183179
11	1.5818201	1.5818201	1.5818201	1.5857230	1.5857230	1.5857230
12	1.5818201	0.4181799	0.4181799	1.5857230	0.4142770	0.4142770
13	0.4181799	0.4181799	0.4181799	0.4142770	0.4142770	0.4142770
14	0.3052577	1.3645144	0.0000000	0.2816821	1.3513493	0.0000000
15	2.0000000	1.5000000	1.0000000	2.0000000	1.5000000	1.0000000
16	1.6354856	1.3052577	1.0000000	1.6486507	1.2816821	1.0000000
17	1.3052577	1.0000000	0.3645144	1.2816821	1.0000000	0.3513493
18	1.0000000	1.6354856	1.3052577	1.0000000	1.6486507	1.2816821
19	0.5818201	1.4181799	0.5818201	0.5857230	1.4142770	0.5857230
20	1.0000000	0.0000000	1.5000000	1.0000000	0.0000000	1.5000000
21	1.5818201	0.4181799	1.5818201	1.5857230	0.4142770	1.5857230
22	1.5000000	1.0000000	2.0000000	1.5000000	1.0000000	2.0000000
23	1.6947423	0.6354856	2.0000000	1.7183179	0.6486507	2.0000000
24	0.6354856	0.0000000	0.3052577	0.6486507	0.0000000	0.2816821
25	0.4181799	1.5818201	1.5818201	0.4142770	1.5857230	1.5857230
26	0.3645144	1.3052577	1.0000000	0.3513493	1.2816821	1.0000000
27	1.6354856	0.6947423	1.0000000	1.6486507	0.7183179	1.0000000
28	0.6947423	1.0000000	0.3645144	0.7183179	1.0000000	0.3513493
29	1.4181799	1.4181799	1.4181799	1.4142770	1.4142770	1.4142770
30	0.6354856	0.0000000	1.6947423	0.6486507	0.0000000	1.7183179
31	2.0000000	1.6947423	1.3645144	2.0000000	1.7183179	1.3513493
32	1.3052577	1.0000000	1.6354856	1.2816821	1.0000000	1.6486507
33	0.5000000	1.0000000	0.0000000	0.5000000	1.0000000	0.0000000
34	0.3052577	0.6354856	0.0000000	0.2816821	0.6486507	0.0000000
35	0.3645144	0.6947423	1.0000000	0.3513493	0.7183179	1.0000000
36	1.4181799	0.5818201	0.5818201	1.4142770	0.5857230	0.5857230
37	0.5818201	0.5818201	0.5818201	0.5857230	0.5857230	0.5857230
38	0.5818201	1.4181799	1.4181799	0.5857230	1.4142770	1.4142770
39	2.0000000	0.3052577	1.3645144	2.0000000	0.2816821	1.3513493
40	0.4181799	0.4181799	1.5818201	0.4142770	0.4142770	1.5857230
41	2.0000000	0.5000000	1.0000000	2.0000000	0.5000000	1.0000000
42	1.0000000	0.3645144	0.6947423	1.0000000	0.3513493	0.7183179
43	0.6947423	1.0000000	1.6354856	0.7183179	1.0000000	1.6486507
44	1.4181799	0.5818201	1.4181799	1.4142770	0.5857230	1.4142770
45	0.5818201	0.5818201	1.4181799	0.5857230	0.5857230	1.4142770
46	1.0000000	0.3645144	1.3052577	1.0000000	0.3513493	1.2816821