Complementary Relationship
between the Golden Ratio and the Silver Ratio in the Three-Dimensional Space &
"Golden Transformation" and "Silver Transformation"
applied to Duals of Semi-regular Convex Polyhedra

September 1, 2010
Hiroaki Kimpara

1. Preface

The Golden Ratio $\mathbf{1} : \dfrac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}}$ and the Silver Ratio $\mathbf{1} : \sqrt{2}$ serve as the fundamental ratio in the shape of objects and they have so far been considered independent of each other. However, the author has found out, through application of the "Golden Transformation" of rhombic dodecahedron and the "Silver Transformation" of rhombic triacontahedron, that these two ratios are closely related and complementary to each other in the 3-demensional space.

Duals of semi-regular polyhedra are classified into 5 patterns. There exist 2 types of polyhedra in each of 5 patterns. In case of the rhombic polyhedra, they are the dodecahedron and the triacontahedron. In case of the trapezoidal polyhedral, they are the icosidodecahedron and the hexecontahedron. In case of the pentagonal polyhedra, they are icositetrahedron and the hexecontahedron. Of these 2 types, the one with the lower number of faces is based on the Silver Ratio and the one with the higher number of faces is based on the Golden Ratio.

The "Golden Transformation" practically means to replace each face of: (1) the rhombic dodecahedron with the face of the rhombic triacontahedron, (2) trapezoidal icosidodecahedron with the face of the trapezoidal hexecontahedron, and (3) the pentagonal icositetrahedron with the face of the pentagonal hexecontahedron. These replacing faces are all partitioned into triangles. Similarly, the "Silver Transformation" means to replace each face of: (1) the rhombic triacontahedron with the face of the rhombic dodecahedron, (2) trapezoidal hexecontahedron with the face of the trapezoidal icosidodecahedron, and (3) the pentagonal hexecontahedron with the face of the pentagonal icositetrahedron. Again, the replacing faces are all divided into triangles.

While the aforementioned rhombic polyhedra both belong to Pattern Ⅱ, the trapezoidal polyhedra belong to Pattern Ⅳ. On the other hand, the pentagonal polyhedra belong to Pattern Ⅴ. Accordingly, the author conducted "Golden Transformation" and "Silver Transformation" of the trapezoidal polyhedra and the pentagonal polyhedra, too, and looked into their results. It is to be noted that the "Golden Transformation" and the "Silver Transformation" cannot be applied to duals belonging to Pattern Ⅰ and Ⅲ because faces of these polyhedra are all triangles.

The outcome of this study is presented in this paper.

2. Golden Ratio and Silver Ratio found in the Natural and Human Worlds

The Golden Ratio and Silver Ratio are found in various aspects of the natural and human worlds. Their actual examples are shown in Table 1. (See References [1] and [2].) This comparative table suggests that the Silver Ratio serves as the basic and primordial ratio not only in the visible macro world but also in the microscopic realms invisible to the naked eye.

3. Polyhedra based on the Silver Ratio and Those based on the Golden Ratio

Among regular polyhedra, the tetrahedron, cube, and octahedron are based on the Silver Ratio, whereas the dodecahedron and icosahedron are predicated on the Golden Ratio. Also, the cub-octahedron and rhombic dodecahedron mentioned in Table 1 are based on the Silver Ratio. The reasons are mentioned in Table 2 below.

	Reasons
Regular tetrahedron	The dihedral angle is 70°31'43". This is the same as the acute angle of the silver rhombus with a ratio of the short diagonal to the long diagonal being equal to the Silver Ratio. The obtuse angle of the isosceles triangle made by connecting the center of the gravity with any two vertices is 109°28'. This is the same as the obtuse angle of the silver rhombus.
Cube	Any two faces in parallel with each other are considered. If 2 ends of the upper edge of one face are connected with 2 ends of the lower edge of the other face, a polygon is created. This polygon is the "Silver Rectangle" with a ratio of its 2 different edges being equal to the Silver Ratio.
Regular octahedron	Six vertices are the vertices of three squares that intersect one another at the right angle at the center. The ratio of any edge of the square to its diagonal is equal to the Silver Ratio. The dihedral angle is 109°28'16". This is the same as the obtuse angle of the silver rhombus with a ratio of the short diagonal to the long diagonal being equal to the Silver Ratio.
Regular dodecahedron	Twelve centers of the 12 pentagons are the vertices of 3 rectangles that intersect one another at the right angle at the center. These 3 congruent rectangles are all "Golden Rectangle" with a ratio of 2 different edges being equal to the Golden ratio. The dihedral angle is 116°33'54". This is the same as the obtuse angle of the golden rhombus with a ratio of the short diagonal to the long diagonal being equal to the Golden Ratio.
Cub-octahedron	The cub-octahedron is made by connecting the middle point of each edge of either the cube or the regular octahedron. These solids are both based on the Silver Ratio.

Regular icosahedron	Twelve vertices are the vertices of three rectangles that intersect one another at the right angle at the center. These 3 congruent rectangles are all "Golden Rectangle" with a ratio of 2 different edges being equal to the Golden Ratio.
Rhombic dodecahedron	This solid is composed of 12 congruent rhombuses. The ratio of its short diagonal to long diagonal is equal to the Silver Ratio.

Table 2: Reasons why selected polyhedra are based on either the Golden Ratio or the Silver Ratio

4. Application of "Golden Transformation" & "Silver Transformation" to Duals of Quasi-regular Polyhedra

Duals of semi-regular polyhedra are classified into the following 5 patterns:

	Duals of Quasi-regular Polyhedra	
Pattern I	*p-kis F-hedron*	Dual of quasi-regular polyhedron [p,2q,2q], made of regular F-hedron (p,q).
II	Rhombic E-hedron	Dual of quasi-regular polyhedron [p,q,p,q] with E being the number of edges of regular polyhedron (p,q).
III	Hexakis F-hedron	Dual of quasi-regular polyhedron [4,2p,2q]. Faces are scalene triangles and they alternately become mirrored images of adjacent ones, with (p,q) being a regular F-hedron and p being equal to 3.
IV	Trapezoidal $2E$-hedron	Dual of quasi-regular polyhedron [3,4,4,4] or [3,4,5,4]. Faces are trapezoids. This corresponds to a case of Pattern III where triangles on both sides are on the same plane against a line segment which connects an original vertex with a point at a certain distance measured along a perpendicular line from the center of the face.
V	Pentagonal $2E$-hedron	Dual of quasi-regular polyhedron [3,3,3,3,4] or [3,3,3,3,5]. Faces are irregular pentagons. Two of their edges holding the vertex with an acute angle between them are isometric and 3 other edges are also isometric. Four other interior angles at the other 4 vertices are also the same.

Table 3: Five Patterns of Duals of Quasi-regular Polyhedra

4.1 Rhombic Polyhedra

Among these polyhedral duals, the rhombic triacontahedron and the rhombic dodecahedron belong to Pattern II. The former is composed of 30 golden rhombuses and the latter 12 silver rhombuses. New solids can be created by mutually replacing these rhombuses which are partitioned into two. There are two ways of halving the rhombuses. One is the partition by the long diagonal and the other by the short diagonal.

4.1.1 Polyhedra created by Silver Transformation

First of all, each face of the rhombic triacontahedron is replaced with 2 isosceles triangles arising from the partition of the silver rhombus by the long diagonal. This process is the aforementioned Silver Transformation (abbreviated as S.T.) Then, a nonconvex polyhedron composed of 60 congruent isosceles triangles with obtuse angles is created. (Refer to Figure 1a.) This new polyhedron having its own circumsphere is considered to be of the same kind as the triakis icosahedron. For this reason, it is tentatively named "Triakis Icosahedron derived from S.T. [1st kind]". This solid can be seen as an icosahedron with a trigonal pyramid covering each face and the dihedral angle of the pyramid against its base is 35˚15'52". The length of three edges of this triangle is at a ratio of $2\sqrt{2} : \sqrt{3} : \sqrt{3}$. This polyhedron represents stellation of the regular icosahedron. The dihedral angle created by the diagonal of the silver rhombus is 151°16'51".

Replacement of each face of the rhombic triacontahedron with 2 isosceles triangles arising from the partition of the silver rhombus by the short diagonal results in the creation of a convex polyhedron consisting of 60 congruent, isosceles triangles with acute angles. (Refer to Fig. 1b.) This new polyhedron having its own inscribed sphere is considered to be of the same kind as the pentakis dodecahedron. So, it is provisionally named "Pentakis Dodecahedron derived from S.T. [1st kind]". It can be seen as a dodecahedron with a pentagonal pyramid covering each face, with its dihedral angle being 31˚19'03". The length of 3 edges of this triangle is at a ratio of $\sqrt{3} : \sqrt{3} : 2$.

Fig. 1a: Triakis Icosahedron derived from S.T. [1st kind] (top view)

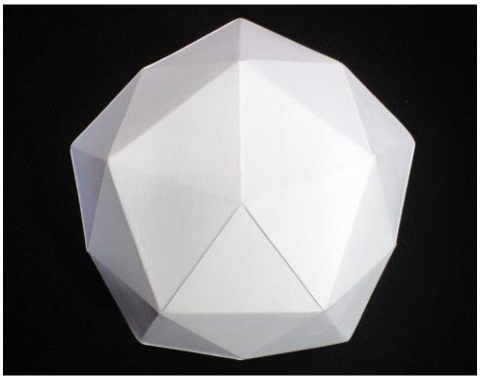

Fig. 1b: Pentakis Dodecahedron derived from S.T [1st kind] (top view)

In case of the "Triakis Icosahedron derived from S.T. [1st kind]", vertices of 20 triangular pyramids are all outward-directed. If they are directed inward, a new nonconvex polyhedron is created. (Refer to Fig. 1c). It represents further stellation of "Triakis Icosahedron derived from S.T. [1st kind]" and, therefore, it is tentatively named "Triakis Icosahedron derived from S.T. [2nd kind]". This is very similar to the Great Dodecahedron (5, 5/2), one of the 4 Kepler–Poinsot polyhedra. The dihedral angle created by the diagonal of the silver rhombus is 67°40'.

In case of the “Pentakis Dodecahedron derived from S.T. [1st kind]”, vertices of 12 pentagonal pyramids are all outward-directed. If these verities are directed inward, a new nonconvex polyhedron is also created. (Refer to Fig. 1d). This solid is provisionally named “Pentakis Dodecahedron derived from S.T. [2nd kind]”. The dihedral angle created by the diagonal of the silver rhombus is the right angle. This is considered a very singular phenomenon.

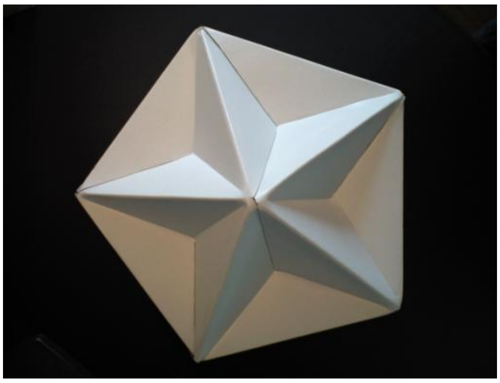

Fig. 1c: Triakis Icosahedron derived from from S.T. [2nd kind] (top view)

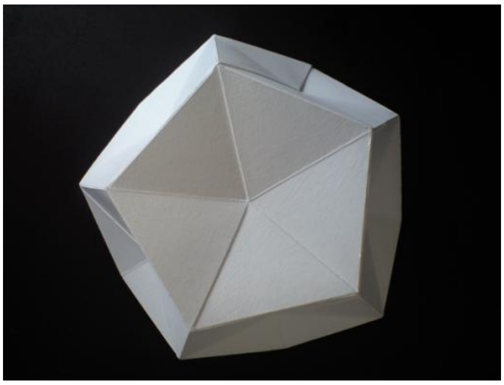

Fig. 1d: Pentakis Dodecahedron derived S.T. [2nd kind] (top view)

4.1.2 <u>Polyhedra created by Golden Transformation</u>

Then, each face of the rhombic dodecahedron is replaced with 2 isosceles triangles arising from the partition of the golden rhombus by the long diagonal. This process is the aforementioned Golden Transformation (abbreviated as G.T.) and it results in the creation of a convex polyhedron composed of 24 congruent isosceles triangles with obtuse angles. (Refer to Fig. 2a and 2b.) Representing stellation of the regular octahedron, this new solid is considered to be of the same kind as the triakis octahedron. Accordingly, this polyhedron is tentatively named “Triakis Octahedron derived from G.T. [1st kind]”. The length of three edges of this triangle is at a ratio of $1+\sqrt{5}$: $\sqrt{\frac{5+\sqrt{5}}{2}}$: $\sqrt{\frac{5+\sqrt{5}}{2}}$. It can be seen as an octahedron with a trigonal pyramid covering each face and the dihedral angle of the pyramid against its base is 20˚54’19”. The dihedral angle created by the long diagonal of the golden rhombus is 151°16’51” and the other dihedral angle is 144°.

In case of the “Triakis Octahedron derived from G.T. [1st kind]”, vertices of 8 triangular pyramids are all outward-directed. If these verities are directed inward, a new nonconvex polyhedron is created. (Refer to Fig. 2c and 2d). It represents stellation of “Triakis Octahedron derived from G.T. [1st kind]” and, therefore, this solid is tentatively named “Triakis Octahedron derived from G.T. [2nd kind]”. The dihedral angle created by the long diagonal of the golden rhombus is 67°40’.

Replacement of each face of the rhombic dodecahedron with 2 isosceles triangles arising from the partition of the golden rhombus by the short diagonal results in the creation of a nonconvex polyhedron composed of 24 congruent isosceles triangles with acute angles. (Refer to Fig. 2e and 2f.) The length of three edges of this triangle is at a ratio of $2 : \sqrt{\frac{5+\sqrt{5}}{2}} : \sqrt{\frac{5+\sqrt{5}}{2}}$. Representing stellation of the cube, this new solid is considered to be of the same kind as the tetrakis hexahedron. For this reason, it is provisionally named "Tetrakis Hexahedron derived from G.T." This polyhedron can be seen as a cube with a quadrangular pyramid covering each face and the dihedral angle of the pyramid against its base is 51˚49'38". The dihedral angle created by the short diagonal of the golden rhombus is 166˚21'.

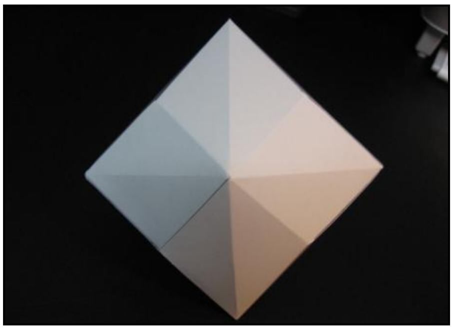

Fig. 2a: "Triakis Octahedron derived from G.T. [1st kind]" (top view)

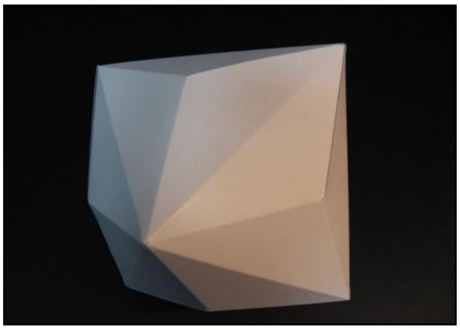

Fig. 2b: "Triakis Octahedron derived from G.T. [1st kind]" (perspective view)

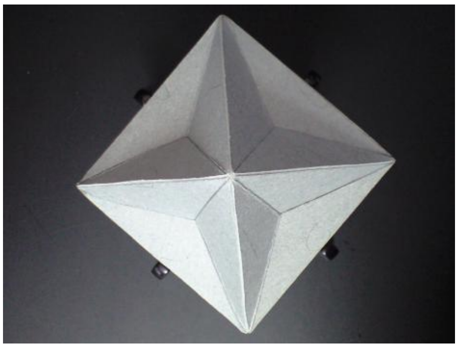

Fig. 2c: Triakis Octahedron derived from G.T. [2nd kind] (top view)

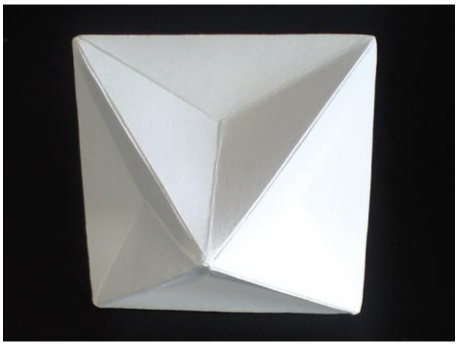

Fig. 2d: Triakis Octahedron derived from G.T. [2nd kind] (perspective view)

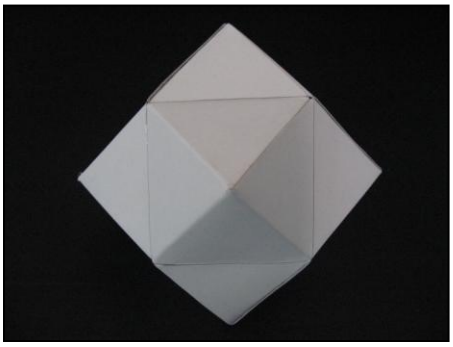

Fig. 2e: Tetrakis Hexahedron derived from G.T. (top view)

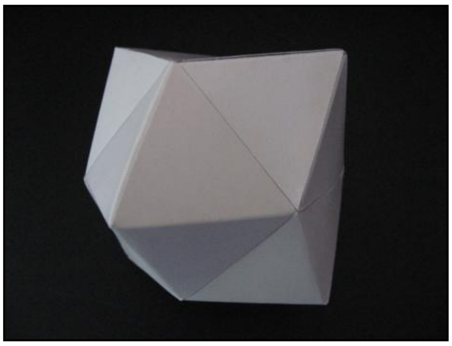

Fig. 2f: Tetrakis Hexahedron derived from G.T. (perspective view)

The aforementioned pentakis dodecahedron, triakis icosahedrons, tetrakis hexahedron, and triakis octahedron are all polyhedral duals classified into Pattern I. This means that **application of the "Golden Transformation" process and the "Silver Transformation" process to the duals belonging to Pattern II results in the creation of new polyhedral duals belonging to Pattern** I.

It was revealed that the convex portion of the "Triakis Octahedron derived from G.T. [1st kind]" with the dihedral angle 151˚16'52" tightly fits into the concave portion of the "Triakis Icosahedron derived from S.T. [1st kind]" with the dihedral angle 151˚16'52". (Refer to Figure 3.) Furthermore, the dihedral angle of the "Triakis Icosahedron derived from S.T. [2nd kind]" being 67˚40', is completely equal to the one of the "Triakis Octahedron derived from G.T. [2nd kind]" being 67˚40'.

This fact strongly suggests that **the Golden Ratio and the Silver Ratio are complimentary to each other in the three-dimensional space**.

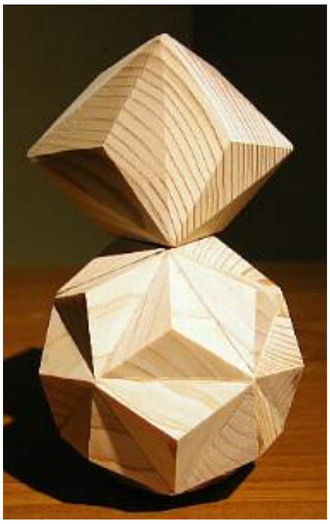

Fig. 3: The convex portion of the "Triakis Octahedron derived from G.T. [1st kind]" with the dihedral angle 151˚16'52" (above) tightly fits into the concave portion of the "Triakis Icosahedron derived from S.T. [1st kind]" with the dihedral angle 151˚16'52" (below).

4.2 Trapezoidal Polyhedra

The dual of the trapezoidal icosidodecahedron is the rhombicuboctahedron and the one of the trapezoidal hexacontahedron is the rhombicosidodecahedron. The originals of these quasi-regular polyhedra are the cube and the regular dodecahedron, respectively. As mentioned earlier in this paper, these regular polyhedra are respectively based on the Silver Ratio and the Golden Ratio. Accordingly, it is presumed that S.T. and G.T. applied to the rhombic polyherda can be applied to the trapezoidal polyhedra, too. On the basis of this perception, the trapezoids of the trapezoidal icosidodecahedron and the trapezoidal hexacontahedron are hereinafter called "Silver Trapezoid" and "Golden Trapezoid", respectively.

There are two ways of partitioning the “Silver Trapezoid” and “Golden Trapezoid” into two. One is to halve them by the diagonal drawn from the vertex with the obtuse angle to its opposite vertex and the other is to divide them by another diagonal connecting the remaining two vertices. In case of the former, two scalene triangles become congruence. In case of the latter, however, two different isosceles triangles are produced. The interior angles of these triangles and the ratio of their respective edges are shown in Table 4 below.

	Silver Trapezoid		Golden Trapezoid	
	Interior angle	Ratio of lengths of 3 edges	Interior angle	Ratio of lengths of 3 edges
Partition by the diagonal drawn from the vertex with the obtuse angle to its opposite vertex	57°38’ 40°47’20” 81°34’40”	1: 1.2929 : 1.5143	59°8’ 33°53’ 86°59’	1 : 1.5393 : 1.7909
Partition by another diagonal connecting the remaining two vertices	Not applicable	Not applicable	118°16’ 30°52’ 30°52’	1: 1: 1.7166
	Not applicable	Not applicable	67°46’ 56°7’ 56°7’	1 : 1 : 1.1152

Table 4: Interior Angles of Triangles and the Ratio of Lengths of Their Respective Edges

Replacement of each face of the trapezoidal hexacontahedron with 2 congruent, scalene triangles arising from bisection of the Silver Trapezoid results in the creation of a beautifully –stellated nonconvex polyhedron consisting of 120 congruent, scalene triangles. (Refer to Fig. 4.) This solid is considered to be of the same kind as hexakis icosahedrons and, accordingly, it is tentatively named the “Hexakis Icosahedron derived from S.T.” It can also be regarded as a dodecahedron with the vertex portion of the Great Dodecahedron (5, 5/2), one of the four Kepler–Poinsot polyhedra, covering each face. It can also be thought of a rhombic triacontahedron with a quadrangular pyramid covering each face and the vertex of such pyramid is directed inward. The “Hexakis Icosahedron derived from S.T.” has three dihedral angles and two of them are 74°57’20” and 116°34’55”.

Then, each face of the trapezoidal icosidodecahedron is replaced with 2 congruent, scalene triangles arising from bisection of the Golden Trapezoid. This results in the creation of a convex polyhedron with 48 faces (Refer to Fig. 5a and 5b). This solid is considered to be of the same kind as the hexakis octahedron and therefore it is provisionally named the “Hexakis Octahedron derived from G.T. [1st kind]”. This polyhedron can be seen as an octahedron with 6 scalene triangles covering each face.

The “Hexakis Octahedron derived from G.T. [1st kind]” also has three dihedral angles and two of them are 170°51’48” and 127°30’40”.

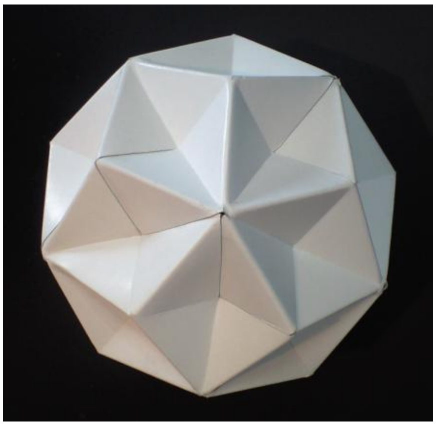

Fig. 4: Hexakis Icosahedron derived from S. T. (top view)

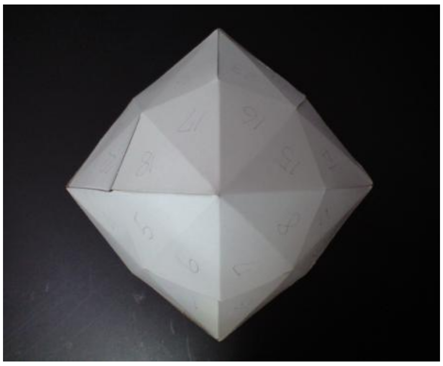

Fig. 5a: Hexakis Octahedron derived from G.T. [1st kind](top view)

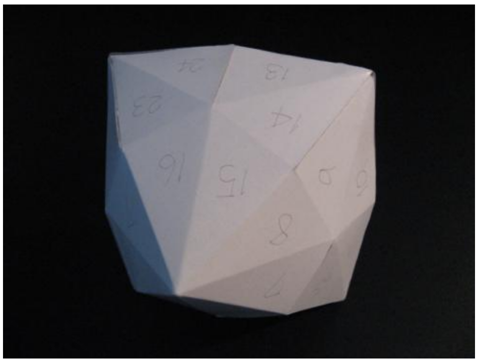

Fig. 5b: Hexakis octahedron based on G.T. [1st kind] (perspective view)

When another partitioning (based on the diagonal connecting two vertices with the acute angle) is used, only G.T. can be applied. The trapezoidal hexecontahedron consists of 12 groups of “Golden Trapezoid” and each group is comprised of 5 “Golden Trapezoids”. Five tips of such trapezoids converge at the center of each group. The tip vertex angle of “Golden Trapezoid” is 67˚46’. This means that the total angle resulting from such convergence is 338˚50’, which is less than 360˚.

On the other hand, the tip vertex angle of “Silver Trapezoid” is 81˚34’44”. If 5 tips of “Silver Trapezoid” get together, the total angle exceeds 360˚. Application of S.T. to the trapezoidal hexecontahedron requires the tip vertex angle to be split. However, the second partitioning method (based on the diagonal connecting two vertices with the acute angle) is unable to do it.

Replacement of each face of the trapezoidal icosidodecahedron with 2 non-congruent, isosceles triangles arising from the aforementioned partition of the Golden Trapezoid results in the creation of a nonconvex polyhedron with 48 faces. It is similar to the above-mentioned hexakis octahedron. For this reason, this new solid is tentatively named the "Hexakis Octahedron derived from G.T. [2nd kind]". It can be also seen as an octahedron with each face composed of 2 kinds of isosceles triangles, 3 triangles each. (Refer to Figures 6a and 6b.)　This solid is featured by the regular, trigonal pyramid located at the center of each face. Three edges of its base make a concave shape.

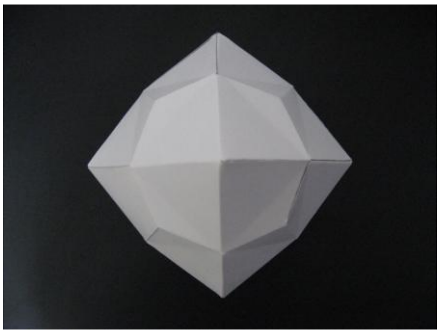

Fig. 6a: Hexakis Octahedron derived from G.T. [2nd kind] (top view)

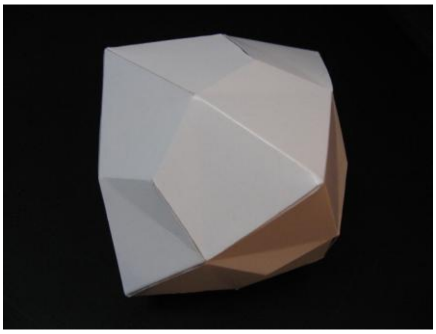

Fig. 6b: Hexakis Octahedron derived from G.T. [2nd kind] (perspective view)

The aforementioned hexakis icosahedron and hexakis octahedron are both polyhedral duals classified into Pattern Ⅲ. **This means that application of G.T. and S.T. to the duals belonging to Pattern Ⅳ results in the creation of new polyhedral duals belonging to Pattern Ⅲ.**

4.3　Pentagonal Polyhedra

The dual of the pentagonal icositetrahedron is the snab cube [3,3,3,3,4] and the one of the pentagonal hexecontahedron is the snab dodecahedron [3,3,3,3,5]. The originals of these quasi-regular polyhedra are the cube and the regular dodecahedron, respectively. As mentioned earlier in this paper, these regular polyhedra are respectively based on the Silver Ratio and the Golden Ratio. Accordingly, it is presumed that S.T. and G.T. applied to the rhombic polyherda can also be applied to the pentagonal polyhedral. On the basis of this perception, the pentagons of the pentagonal icositetrahedron and the pentagonal hexecontahedron are hereinafter called "Silver Pentagon" and "Golden Pentagon", respectively.

Described in Table 5 below are three ways of dividing the "Silver Pentagon" and "Golden Pentagon" into three triangles.

It is to be noted that the applicable transformation method is G.T. alone when the 3rd partitioning method is used. In other word, S.T. is not applicable in this case.

The pentagonal hexecontahedron consists of 12 groups of "Golden Pentagon" and each group is composed of 5 "Golden Pentagons". Five tips of such pentagons converge at the center of each group. The tip vertex angle of "Golden Pentagon" is 67˚28'. This means that the total angle resulting from such convergence is 337˚20', which is less than 360˚.

	Dividing method
1	Division by 2 diagonals drawn from the head vertex (resulting in one isosceles triangle and 2 congruent, scalene triangles)
2	Division by 2 diagonals drawn from the base vertex, either right-hand side or left-hand side (resulting in one isosceles triangle and 2 non-congruent, scalene triangles)
3	Division by 2 diagonals drawn from the vertex between the head and the base, either right-hand side or left-hand side (resulting in 2 non-congruent, isosceles triangles and one scalene triangle)

Table 5: Three Ways of Dividing the "Silver Trapezoid" and "Golden Trapezoid" into 3 Triangles

On the other hand, the tip vertex angle of "Silver Pentagon" is 80˚46'. If 5 tips of "Silver Pentagon" get together, the total angle exceeds 360˚. Application of S.T. to the pentagonal hexecontahedron requires the tip vertex angle to be split. However, the 3rd partitioning method (based on 2 diagonals drawn from the vertex between the head and the base) is unable to do it.

The interior angles of triangles and the ratio of their respective edges are shown in Table 6 below.

Pentagon type / Dividing method	Silver Pentagon		Golden Pentagon	
	Interior angle	Ratio of lengths of 3 edges	Interior angle	Ratio of lengths of 3 edges
No.1 in Table 5 (Division by two diagonals drawn from the head vertex)	26˚16', 38˚55'30" 114˚48'30"	1: 1.4196 : 2.0511	21˚39' 40˚13' 118˚8'	1: 1.7499 : 2.3901
	28˚14' 75˚53' 75˚53'	1: 2.0511 : 2.0511	24˚10' 77˚55' 77˚55"	1: 2.3901: 2.3901
No. 2 in Table 5 (Division by two diagonals drawn from the base vertex, either right-hand side or left-hand side)	26˚16' 38˚55'30" 114˚48'30"	1: 1.4196 : 2.0511	21˚39' 40˚13' 118˚8'	1: 1.7499 : 2.3901
	54˚30' 82˚12'45" 43˚17'15"	1: 1.1870 : 1.4448	45˚49' 46˚59' 87˚12'	1: 1.0192 : 1.3922
	32˚35'45 32˚35'45" 114˚48'30"	1: 1 : 1.6850	30˚56' 30˚56' 118˚8'	1: 1 : 1.7168
No. 3 in Table 5	Not applicable	Not applicable	67˚28' 56˚16' 56˚16'	1: 1 : 1.1128

No. 3 in Table 5 (Division by two diagonals drawn from the vertex between the head and the base, either right-hand side or left-hand side)	Not applicable	Not applicable	30˚56’ 61˚52’ 87˚12’	1: 1.7168 : 1.9473
	Not applicable	Not applicable	30˚56’ 30˚56’ 118˚8’	1: 1 : 1.7168

Table 6: Interior Angles of Triangles and the Ratio of Lengths of Their Respective Edges

The pentagonal icositetrahedron and the pentagonal hexecontahedron both have two distinct forms, i.e. the right-hand system and the left-hand system, which are mirror images of each other. For this reason, it is presumed that polyhedra derived from the application of G.T. and S.T. also have the right-hand system and the left-hand system which are mirror images of each other. If it is to be realized, however, the “Silver Pentagon” and the “Golden Pentagon” must be symmetrically split. This condition can be met only by the first partitioning method.

4.3.1 Division by Two Diagonals drawn from the Head Vertex

Replacement of each face of the pentagonal hexecontahedron with 2 congruent, scalene triangles and one isosceles triangle arising from the aforementioned partition of the Silver Pentagon results in the creation of a beautifully-stellated nonconvex polyhedron with 180 faces. This new polyhedron can be seen as an icosahedron with each face composed of 9 triangles. Therefore, it is tentatively named the “Enneakis Icosahedron [1st kind]”. This solid can also be regarded as a rhombic triacontahedron with each face being concave, composed of 6 triangles. The “Enneakis Icosahedron [1st kind]” has 4 dihedral angles and 3 of them are 156°9’49”, 67°6’7” and 131°18’7”. As presumed above, this solid has the right-hand system and the left-hand system, which are mirror images of each other. (Refer to Fig. 7a.)

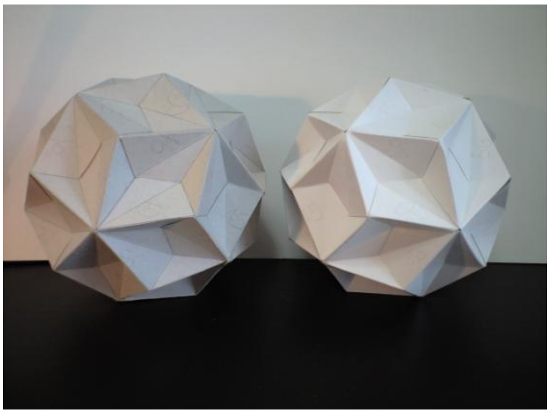

Fig. 7a: Enneakis Icosahedrons [1st kind] (right-hand and left-hand systems)

Fig. 7b: Enneakis Octahedron [1st kind] (right-hand and left-hand systems)

Then, each face of the pentagonal icositetrahedron is replaced with 2 congruent, scalene triangles and one isosceles triangle arising from the division of the Golden Pentagon by 2 diagonals drawn from the head vertex. Such G.T. application results in the creation of a convex polyhedron with 72 faces. This new polyhedron can be seen as an octahedron with each face comprised of 9 triangles. Accordingly, it is provisionally named the "Enneakis Octahedron [1st kind]. (Refer to Figures 8a and 8b.) It has 4 dihedral angles and 3 of them are 156°9'49", 67°6'7" and 131°18'7".

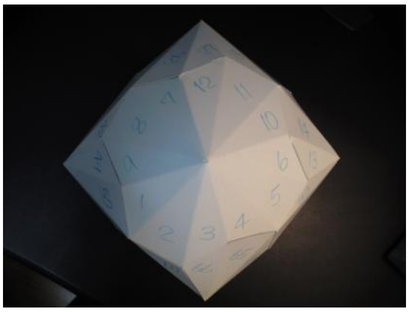

Fig. 8a: Enneakis Octahedron [1st kind] (top view)

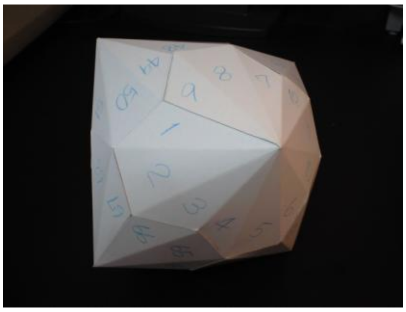

Fig. 8b: Enneakis Octahedron [1st kind] (perspective view)

This solid also has the right-hand system and the left-hand system, which are mirror images of each other. (Refer to Fig. 7b.)

4.3.2 Div. by 2 Diagonals drawn from the Base Vertex, either right-hand side or left-hand side

4.3.2.1 Application of S.T. Process

Each face of the pentagonal hexecontahedron is replaced with one isosceles triangle and 2 non-congruent, scalene triangles arising from the aforementioned partition of the Silver Pentagon. In case the dihedral angle of the long diagonal is convex, it results in the creation of a stellated, non-convex polyhedron with 180 faces which looks more dynamic than the "Enneakis Icosahedron [1st kind]". Like the "Enneakis Icosahedron [1st kind]", this new polyhedron can be seen either as an icosahedron with each face composed of 9 triangles or as a rhombic triacontahedron with each face being concave, composed of 6 triangles. Therefore, it is tentatively named the "Enneakis Icosahedron [2nd kind]". (Refer to Fig. 9.) This solid also has the right-hand system and the left-hand system.

However, a polyhedron of quite a different figure is created in case the dihedral angle of the long diagonal is concave. Like the "Enneakis Icosahedron [1st kind]" and "Enneakis Icosahedron [2nd kind]", it can be regarded as an icosahedron with each face composed of 9 triangles. Accordingly, this solid is provisionally named the "Enneakis Icosahedron [3rd kind]". It is characterized by the trigonal pyramid located at the center of each face, with its tip inward-directed. (Refer to Fig. 10.) It has the right-hand and the left-hand systems, too.

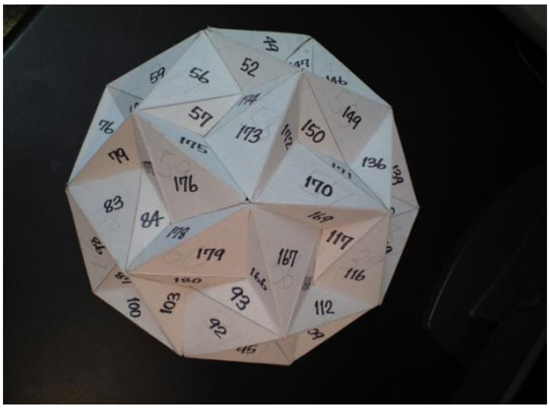

Fig. 9: Enneakis Icosahedron [2nd kind]

Fig. 10: Enneakis Icosahedron [3rd kind]

4.3.2.2 <u>Application of G.T. Process</u>

Each face of the pentagonal icositetrahedron is replaced with one isosceles triangle and 2 non-congruent, scalene triangles arising from the partition of the Golden Pentagon by 2 diagonals drawn from the base vertex, left-hand side. Such G.T. application results in the creation of a polyhedron with 72 faces. Like the “Enneakis Octahedron [1st kind]”, this new polyhedron can be seen as an octahedron with each face comprised 9 triangles. Accordingly, it is tentatively named the “Enneakis Octahedron [2nd kind, L]. (Refer to Fig. 11a and 11b.) The last alphabet “L” denotes that the diagonals are drawn from the left-hand side vertex.

Three congruent, scalene triangles converge at the center of each face of the regular octahedron, which serves as its basic structure and they form a regular, trigonal pyramid. It is a feature of this polyhedron and is one of major differences from the “Enneakis Octahedron [1st kind]”. Three edges of its base make a concave shape. This solid also has the right-hand system and the left-hand system.

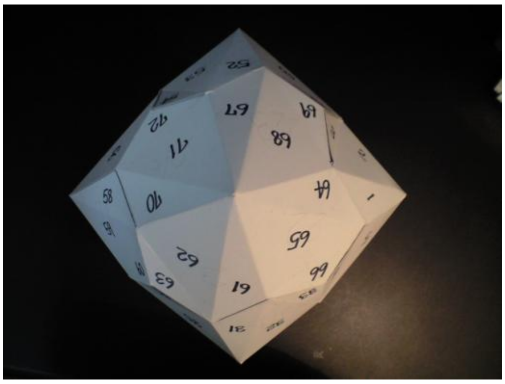

Fig. 11a: Enneakis Octahedron [2nd kind, L] (top view)

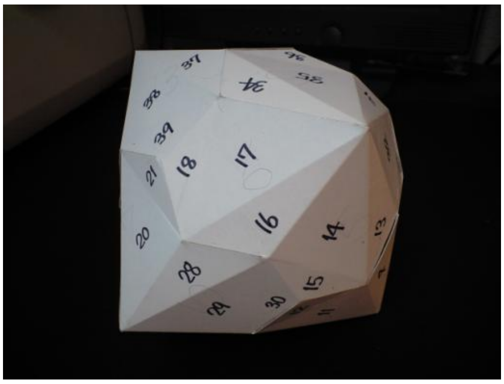

Fig. 11b: Enneakis Octahedron [2nd kind, L] (perspective view)

Then, each face of the pentagonal icositetrahedron is replaced with one isosceles triangle and 2 non-congruent, scalene triangles arising from the division of the Golden Pentagon by 2 diagonals drawn from the base vertex, right-hand side. Such G.T. application results in the creation of a polyhedron with 72 faces. It is basically the same as the "Enneakis Octahedron [2nd kind]". Therefore, it is provisionally named the "Enneakis Octahedron [2nd kind, R]". (Refer to Fig. 12a and 12b.) The last alphabet "R" denotes that the diagonals are drawn from the right-hand side vertex.

Three congruent, scalene triangles diverge from the center of each face just like a propeller. It is a key feature of this polyhedron and is a primary difference from the "Enneakis Octahedron [2nd kind, L]". This solid has the right-hand system and the left-hand system, too.

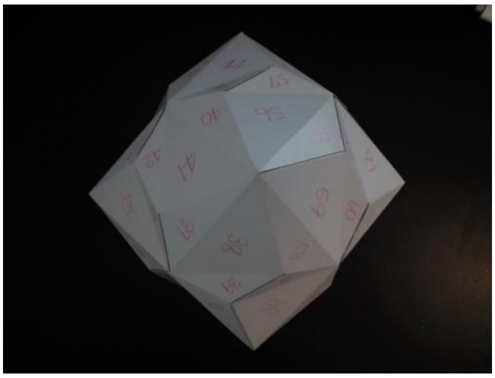

Fig. 12a: Enneakis Octahedron [2nd kind, R] (top view)

Fig. 12b: Enneakis Octahedron [2nd kind, R] (perspective view)

4.3.3 Division by 2 Diagonals drawn from the Vertex between the Head and the Base, either right-hand side or left-hand side

As mentioned earlier in this paper, G.T. process alone is applicable when this partitioning method is used.

Each face of the pentagonal icositetrahedron is replaced with 2 non-congruent, isosceles triangles and one scalene triangle arising from the division of the Golden Pentagon by 2 diagonals drawn from the vertex, right-hand side, between the head and the base. Such G.T. application results in the creation of a polyhedron with 72 faces. Like the aforementioned enneakis octahedra, this new polyhedron can be seen as an octahedron with each face comprised of 9 triangles. On that account, it is tentatively named the "Enneakis Octahedron [3rd kind, R]". (Refer to Fig. 13a and 13b.) The last alphabet "R" stands for "right" as the diagonals are drawn from the right-hand side vertex.

Six vertices of the octahedron are in the form of regular quadrangular pyramid with its base edge making a concave shape. Furthermore, 3 congruent, scalene triangles diverge from the center of each octahedron face just like a propeller. It is a salient feature of this polyhedron. This solid also has the right-hand system and the left-hand system.

Fig. 13a: Enneakis Octahedron[3rd kind, R]
(top view)

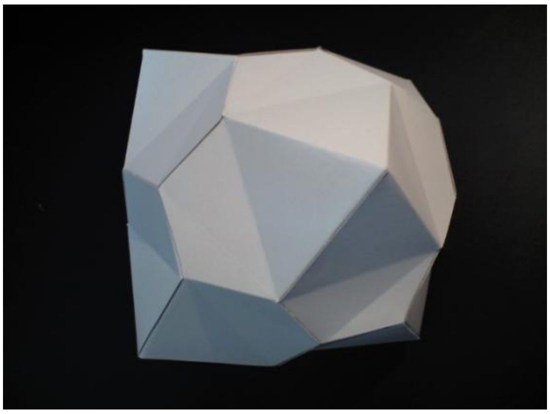

Fig. 13b: Enneakis Octahedron [3^{rd} kind, R]
(perspective view)

Then, each face of the pentagonal hexecontahedron is replaced with 2 congruent, scalene triangles and one isosceles triangle arising from the division of the Golden Pentagon by 2 diagonals drawn from the vertex, left-hand side, between the head and the base. Such G.T. process results in the creation of a polyhedron with 72 faces. It is basically the same as the "Enneakis Octahedron [3rd kind, R]". Therefore, it is provisionally named the "Enneakis Octahedron [3rd kind, L]". The last alphabet "L" denotes that the diagonals are drawn from the left-hand side vertex. Unfortunately, figures of this polyhedron are not available.

Three congruent, scalene triangles converge at the center of each face of the regular octahedron. They serve as its basic structure and respectively form a regular, trigonal pyramid. It is a feature of this polyhedron and is a major difference from the "Enneakis Octahedron [3rd kind, R]". Three edges of its base make a concave shape. This solid has the right-hand system and the left-hand system, too.

Note: Mr. Ikuro Sato of the Research Institute, Miyagi Cancer Center, Japan kindly calculated all of the above-mentioned dihedral angles of polyhedra derived from S.T. and G.T. processes. The author here extends his profound gratitude to him. The author has totally confirmed these calculated data to be accurate through comparison with actual measurement values.

5. Conclusion

It appears that the Golden Ratio and the Silver Ratio have been merely contrasted with each other to date. They also seem to have been considered independent of each other.

As stated in this paper, however, applications of S.T. and G.T. processes to the rhombic polyhedra, trapezoidal polyhedra, and the pentagonal polyhedra have successfully resulted in the creation of new solids with a single closed surface. It was also confirmed that the convex portion of the "Triakis Octahedron derived from G.T. (1st kind)" with the dihedral angle being 151°16'52" tightly fits into the concave portion of the "Triakis Icosahedron derived from S.T. (1st kind)" with the dihedral angle being 151°16'52". (Refer to Figure 3.) Furthermore, the dihedral angle of the "Triakis Icosahedron derived from S.T. (2nd kind)" has been verified to be completely equal to the one of the "Triakis Octahedron derived from G.T. (2nd kind)".

These facts strongly suggest that **the Golden Ratio and the Silver Ratio are complimentary to each other in the three-dimensional space**.

The dihedral angles were also calculated for the new solids derived from S.T. and G.T. applications to the trapezoidal polyhedra and the pentagonal polyhedra. Different from the case of rhombic polyhedron, however, none of them were in good agreement.

On the other hand, it is already known that the golden rectangle based on the Golden Ratio contains cubes infinitely. In other words, this rectangle can be totally filled up with cubes that are put into it infinitely while being reduced in size at a constant ratio. Namely, a smaller golden rectangle is invariably left over in this rectangle even if the biggest cube is removed from it. In a word, it is a rectangle that is diagrammatically tied up with the cube under the optimum relationship. (See Reference [5].) The Silver Ratio is inherent in the cube. Accordingly, this notable characteristic of the golden rectangle is also considered to illustrate the mutually complimentary relationship between these two ratios.

References

[1] H.S.M. Coxecter, Regular Polytopes (third edition), Dover Publications, 1973

[2] M.J.Wenninger, Polyhedron Models, Cambridge University Press, 1971

[3] M.J.Wenninger, Dual Models, Cambridge University Press, 1983

[4] A. Beutelspacher and B. Petri, Golden Section: Nature, Mathematical Principle, and Art,

[5] T. Yohroh, Yuinohron (views on brain), Seido Publications, 2006

[6] Y. Watanabe, Mystery on the Golden Ratio: in search of the Laws on Beauty, Kagakudojin Publications, 2007

[7] K. Akiyama, Mysterious Ancient Figures finally clarified, Cosmo Two-One Publications, 2006

[8] S. Sakurai, Mathematics of Snow, Moon, and Flowers, Shohdensha Publications, 2006